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On a class of solutions of the sine-Gordon equation

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Abstract
In this paper, we consider solutions of the sine-Gordon equation $\frac{\partial^2 f}{\partial\xi\partial\eta} - \sin f = 0$ that are nonlinear analogs of solutions $\sin\left(\alpha + \lambda\xi - \frac{1}{\lambda}\eta\right)$ of the Klein–Gordon equation $\frac{\partial^2 f}{\partial\xi\partial\eta} - f = 0$. Just like the latter may be combined together by means of the Fourier integral to form more complicated solutions of the Klein–Gordon equation, the former may also interact to produce more complicated solutions of the sine-Gordon equation by means of a nonlinear analog of the Fourier integral also derived in the paper.

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(Some figures in this article are in colour only in the electronic version)

1. Sine-Gordon equation: soliton and breather solutions of the sine-Gordon equation

At the dawn of quantum mechanics, a *wave* was defined as a process that exhibits interference and diffraction which require that a wave be spread out through space–time. At the same time, a *particle* was defined to be a process that does not exhibit such behavior and can be described by a motion of a point in the space–time. Although the Hamilton–Jacobi equations provide unified mathematical description of both wave and particle motions, physically the two concepts have been always distinguished. Even in quantum mechanics, the wave and particle aspects of the motion of a wave particle are studied separately and then combined together to describe the corresponding physical process. To much surprise the discovery of solitons provided examples of particle-like solutions to the equations derived to describe wave-like behavior. In [Kov1, Kov2, Kov3, Kov4] and references therein, the author and his collaborators showed that at least some of such equations possess not only purely wave-like and purely particle-like solutions but also solutions that combine the wave-like behavior of linear waves with the particle-like behavior of solitons at the same–time. In this paper, we construct and study wave-particle solutions for the sine-Gordon equation

$$\frac{\partial^2 f}{\partial t^2} - \frac{\partial^2 f}{\partial x^2} + \sin f = 0, \qquad (\mathrm{sG_{tx}})$$

which can be also written as

$$\frac{\partial^2 f}{\partial\xi\,\partial\eta} - \sin f = 0, \qquad (\mathrm{sG}_{\xi\eta})$$

where $x = \eta+\xi, \quad t = \eta-\xi$. We will show that the wave–particle solutions of the sine-Gordon equation exhibit rather interesting properties not seen before.

The *N*-soliton solutions of the sine-Gordon equation are given by formula

$$f(x,t) = -2\mathrm{i}\ln\frac{\det(1+g)}{\det(1-g)} \quad (\mathrm{mod}\, 2\pi), \qquad (1.1a)$$

where g is an $M \times M$ matrix with entries

$$g_{mn} = \frac{c_n}{\mu_n+\mu_m}\,\mathrm{e}^{(\mu_n+\mu_m)\xi+\frac{1}{4}(\frac{1}{\mu_n}+\frac{1}{\mu_m})\eta}, \qquad m,n = 1,2,\ldots,M, \qquad (1.1b)$$

and $\mu_n,\ c_n, n = 1,2,\ldots,M$, are some constants correspondingly called *spectral parameters* and *charges*. The details of the derivation of (1.1) are given in [Nov1] as are the simplest solutions of the sine-Gordon equation which we now review albeit with the terminology somewhat more up-to-date than the terminology of [Nov1].

The simplest complex-valued solution of the sine-Gordon equation

$$f(x,t) = -2\mathrm{i}\ln\frac{2\mu + c\,\mathrm{e}^{2\mu\xi+\frac{1}{2\mu}\eta}}{2\mu - c\,\mathrm{e}^{2\mu\xi+\frac{1}{2\mu}\eta}} \quad (\mathrm{mod}\,2\pi) \qquad (1.2)$$

obtained by taking $M = 1$ in (1.1) is called a *soliton*. Formulas (1.1) are often viewed as nonlinear superposition of M solitons with spectral parameters $\mu_1,\ldots,\mu_M$ and charges $c_1,\ldots,c_M$. For each soliton, one may construct an *antisoliton* by changing the sign of c_1; the nonlinear superposition of a soliton and its antisoliton is zero. We shall call special solitons

$$f = \mp 2\mathrm{i}\ln\frac{1+\mathrm{e}^{2\left(\kappa\alpha+\kappa\xi+\frac{1}{4\kappa}\eta\right)}\cos\varphi + \mathrm{i}\mathrm{e}^{2\left(\kappa\alpha+\kappa\xi+\frac{1}{4\kappa}\eta\right)}\sin\varphi}{1-\mathrm{e}^{2\left(\kappa\alpha+\kappa\xi+\frac{1}{4\kappa}\eta\right)}\cos\varphi - \mathrm{i}\mathrm{e}^{2\left(\kappa\alpha+\kappa\xi+\frac{1}{4\kappa}\eta\right)}\sin\varphi} (\mathrm{mod}\,2\pi) \qquad (1.3a)$$

obtained from (1.1) by taking

$$M = 1, \qquad \mu_1 = \kappa > 0, \qquad c_1 = \pm 2\kappa\,\mathrm{e}^{2\kappa\alpha+\varphi\mathrm{i}}, \qquad \kappa\alpha \in \mathbb{R}, \qquad \varphi \in [0,\pi), \qquad (1.3b)$$

correspondingly *kinks* (if the upper signs are taken) and *antikinks* (if the lower signs are taken). Unless $\varphi = 0$, kinks and antikinks are nonsingular.

We shall call the real nonsingular solutions

$$\begin{aligned} f &= \pm 4\arctan\,\mathrm{e}^{2\left(\kappa\alpha+\kappa\xi+\frac{1}{4\kappa}\eta\right)} \quad (\mathrm{mod}\,2\pi) \\ &= \pm 4\arctan\,\mathrm{e}^{2\kappa\alpha+0.5(\sqrt{\sigma^2+1}\,x-\sigma t)} \quad (\mathrm{mod}\,2\pi), \end{aligned} \qquad (1.4a)$$

obtained by taking $\varphi = \frac{\pi}{2}$ in (1.3) or

$$M = 1, \qquad \mu_1 = \kappa > 0, \qquad c_1 = \pm 2\mathrm{i}\kappa\,\mathrm{e}^{2\kappa\alpha}, \qquad \kappa\alpha \in \mathbb{R}, \qquad \sigma = \frac{4\kappa^2-1}{4\kappa} \qquad (1.4b)$$

in (1.2), correspondingly *real kinks* (if the upper signs are taken) and *real antikinks* (if the lower signs are taken). Snapshots of a real kink and a real antikink are shown in figure 1. Note that one may go from a real kink to its antikink and vice versa not only by alternating between the upper to lower signs but also by changing the sign of φ i.e. going from $\varphi = \frac{\pi}{2}$ to $\varphi = -\frac{\pi}{2}$ and vice versa.

As $\kappa \to +\infty$ and α stays fixed or $\kappa \to 0$ and $\alpha = \frac{\widetilde{\alpha}}{4\kappa^2}$ with fixed $\widetilde{\alpha}$, the real kink and real antikink degenerate into weak solutions of the sine-Gordon equation

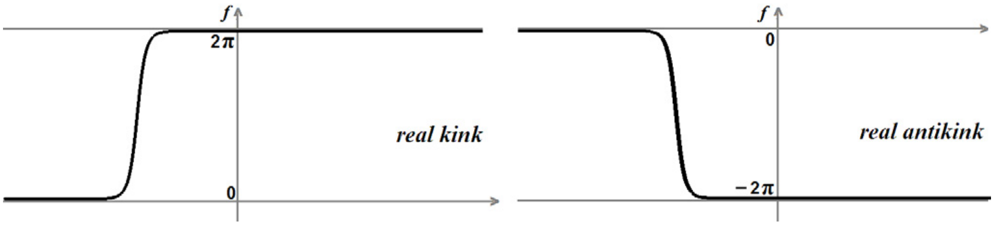

Figure 1. Snapshots of a real kink and a real antikink.

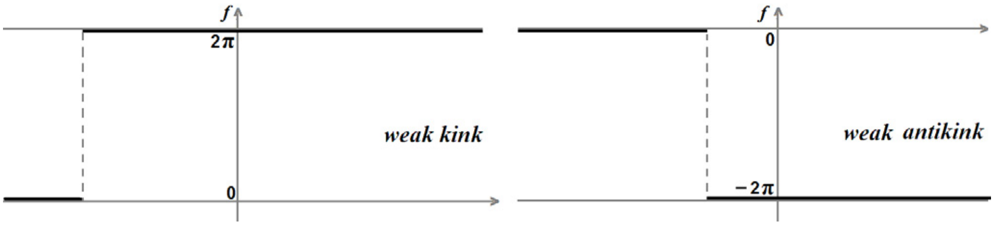

Figure 2. Snapshots of a weak kink and a weak antikink.

$$f = \begin{cases} 0, & \text{if} \quad \alpha + \xi = \alpha + \dfrac{x-t}{2} < 0 \\ \pm 2\pi, & \text{if} \quad \alpha + \xi = \alpha + \dfrac{x-t}{2} > 0 \end{cases} \pmod{2\pi}, \tag{1.5a}$$

$$f = \begin{cases} 0, & \text{if} \quad \widetilde{\alpha} + \eta = \widetilde{\alpha} + \dfrac{x+t}{2} < 0 \\ \pm 2\pi, & \text{if} \quad \widetilde{\alpha} + \eta = \alpha + \dfrac{x+t}{2} > 0 \end{cases} \pmod{2\pi}, \tag{1.5b}$$

that we call correspondingly *weak kinks* (if the upper signs are taken) and *weak antikinks* (if the lower signs are taken). Snapshots of a weak kink and a weak antikink are shown in figure 2

The nonlinear superposition of $\widetilde{M}$ weak kinks and antikinks amounts to plain addition/subtraction and is given by a sum $f_{w1}(x-t) + f_{w2}(x+t)$ of two step functions $f_{w1}(x-t)$, $f_{w2}(x+t)$, each attaining values from the set $0, \pm 2\pi, \pm 4\pi, \pm 6\pi, \ldots$ of multiples of 2π with the first function $f_{w1}(x-t)$ being a function of $x-t$ only while the second function $f_{w2}(x+t)$ is a function of $x+t$ only. Each of the two functions satisfies $\sin f = 0$ and is a weak solution of $\frac{\partial^2 f}{\partial t^2} - \frac{\partial^2 f}{\partial x^2} = 0$, and thus may be viewed as a weak solution of the sine-Gordon equation. The nonlinear superposition of M solitons and $\widetilde{M}$ weak kinks and antikinks is given by $f = f_{\text{sol}} + f_{w1}(x-t) + f_{w2}(x+t)$ with $f = f_{\text{sol}}$ given by (1.1) and $f_{w1}(x-t)$, $f_{w2}(x+t)$ being as just described; the proof is a specific case of the proof of formula (2.3) given at the end of appendix A.

We shall call the imaginary singular solutions

$$\begin{aligned} f &= \pm 2\mathrm{i} \ln \tanh \left(\kappa\alpha + \kappa\xi + \frac{1}{4\kappa}\eta \right) \pmod{2\pi} \\ &= \pm 2\mathrm{i} \ln \tanh \frac{2\kappa\alpha + \sqrt{\sigma^2+1}\, x - \sigma t}{2} \pmod{2\pi}, \end{aligned} \tag{1.6a}$$

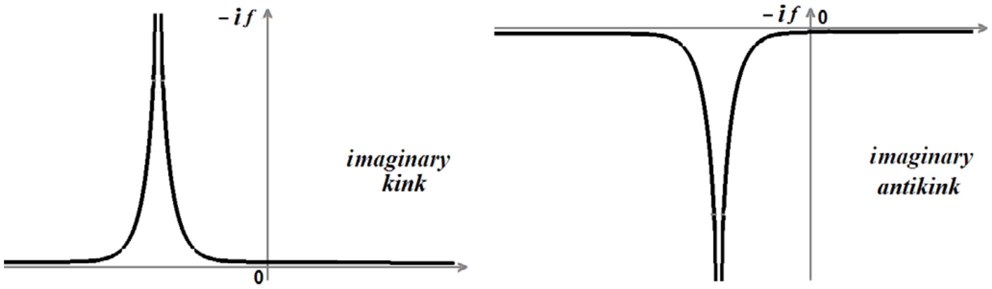

Figure 3. Snapshots of the imaginary parts of an imaginary kink and an imaginary antikink; the real parts are zero.

obtained by taking $\varphi = 0$ in (1.3) or

$$M = 1, \qquad \mu_1 = \kappa > 0, \qquad c_1 = \pm 2\kappa\, \mathrm{e}^{2\kappa\alpha}, \qquad \kappa\alpha \in \mathbb{R}, \qquad \sigma = \frac{4\kappa^2 - 1}{4\kappa} \tag{1.6b}$$

in (1.2), correspondingly *imaginary kinks* (if the upper signs are taken) and *imaginary antikinks* (if the lower signs are taken). Note that one may go from a imaginary kink to its antikink and vice versa not only by alternating between the upper and lower signs but also by changing the value of φ from $\varphi = 0$ to $\varphi = \pi$ and vice versa. A snapshot of imaginary parts of a breather and anti-breather is shown in figure 3.

The real nonsingular solutions of the sine-Gordon equation

$$f = \mp 2\mathrm{i} \ln \frac{\lambda \cosh\left[\kappa\left(2\beta + 2\xi + \frac{1}{2(\kappa^2+\lambda^2)}\eta\right) + \mathrm{i}\varphi\right] + \mathrm{i}\kappa \sin\lambda\left(2\gamma + 2\xi - \frac{1}{2(\kappa^2+\lambda^2)}\eta\right)}{\lambda \cosh\left[\kappa\left(2\beta + 2\xi + \frac{1}{2(\kappa^2+\lambda^2)}\eta\right) + \mathrm{i}\varphi\right] - \mathrm{i}\kappa \sin\lambda\left(2\gamma + 2\xi - \frac{1}{2(\kappa^2+\lambda^2)}\eta\right)} \pmod{2\pi} \tag{1.7a}$$

obtained from (1.1) by taking

$$\begin{aligned} &M = 2, \qquad \mu_1 = \kappa + \mathrm{i}\lambda, \qquad \mu_2 = \kappa - \mathrm{i}\lambda, \\ &c_1 = \pm\frac{2\kappa(\kappa + \mathrm{i}\lambda)}{\lambda}\, \mathrm{e}^{2(\kappa\beta + \lambda\gamma\,\mathrm{i}) + \varphi\,\mathrm{i}}, \qquad c_2 = \mp\frac{2\kappa(\kappa - \mathrm{i}\lambda)}{\lambda}\, \mathrm{e}^{2(\kappa\beta - \lambda\gamma\mathrm{i}) + \varphi\,\mathrm{i}}, \\ &\kappa > 0, \qquad \lambda > 0, \qquad \kappa\beta \in \mathbb{R}, \qquad \lambda\gamma \in \mathbb{R}, \qquad \varphi \in [0, \pi), \end{aligned} \tag{1.7b}$$

are called correspondingly *breathers* (if the upper signs are taken) and *antibreathers* (if the lower signs are taken).

The special real case of (1.7)

$$\begin{aligned} f &= \mp 2\mathrm{i} \ln \frac{\lambda \cosh\kappa\left(2\beta + 2\xi + \frac{1}{2(\kappa^2+\lambda^2)}\eta\right) + \mathrm{i}\kappa \sin\lambda\left(2\gamma + 2\xi - \frac{1}{2(\kappa^2+\lambda^2)}\eta\right)}{\lambda \cosh\kappa\left(2\beta + 2\xi + \frac{1}{2(\kappa^2+\lambda^2)}\eta\right) - \mathrm{i}\kappa \sin\lambda\left(2\gamma + 2\xi - \frac{1}{2(\kappa^2+\lambda^2)}\eta\right)} \pmod{2\pi} \\ &= \pm 4 \arctan \frac{\kappa \sin\lambda\left(2\gamma + 2\xi - \frac{1}{2(\kappa^2+\lambda^2)}\eta\right)}{\lambda\cosh\kappa\left(2\beta + 2\xi + \frac{1}{2(\kappa^2+\lambda^2)}\eta\right)} \pmod{2\pi} \\ &= \pm 4 \arctan \frac{\kappa \sin\left(2\lambda\gamma + \frac{\lambda\omega}{\sqrt{\lambda^2+\kappa^2}}x - \frac{\lambda\sqrt{\omega^2+1}}{\sqrt{\lambda^2+\kappa^2}}t\right)}{\lambda\cosh\left(2\kappa\beta + \frac{\kappa\sqrt{\omega^2+1}}{\sqrt{\lambda^2+\kappa^2}}x - \frac{\kappa\omega}{\sqrt{\lambda^2+\kappa^2}}t\right)} \pmod{2\pi}, \end{aligned} \tag{1.8}$$

where $\omega = \frac{4\kappa^2+4\lambda^2-1}{4\sqrt{\kappa^2+\lambda^2}}$, called correspondingly *real breathers* (if the upper signs are taken) and *real antibreathers* (if the lower signs are taken), is obtained by taking $\varphi = 0$ in (1.7). Snapshots of a real breather and antibreather are shown in figure 4.

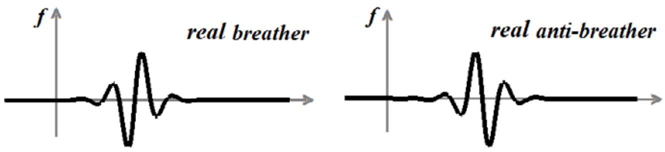

Figure 4. Snapshots of a real breather and a real antibreather.

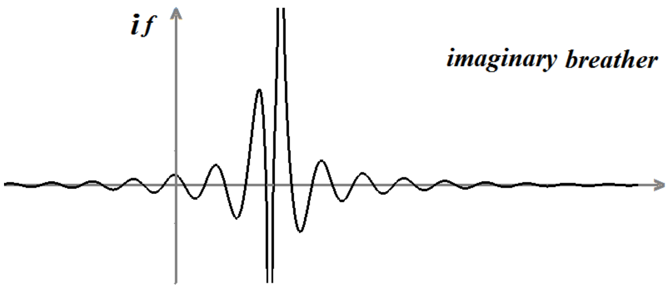

Figure 5. Snapshots of the imaginary parts of an imaginary breather; the real part is zero.

The only singular breathers

$$f = \mp 2\mathrm{i}\ln\frac{\lambda\sinh\kappa\left(2\beta+2\xi+\frac{1}{2(\kappa^2+\lambda^2)}\eta\right)+\kappa\sin\lambda\left(2\gamma+2\xi-\frac{1}{2(\kappa^2+\lambda^2)}\eta\right)}{\lambda\sinh\kappa\left(2\beta+2\xi+\frac{1}{2(\kappa^2+\lambda^2)}\eta\right)-\kappa\sin\lambda\left(2\gamma+2\xi-\frac{1}{2(\kappa^2+\lambda^2)}\eta\right)} \pmod{2\pi}$$

$$= \mp 2\mathrm{i}\ln\frac{\lambda\sinh\left(2\kappa\beta+\frac{\kappa\sqrt{\omega^2+1}}{\sqrt{\lambda^2+\kappa^2}}x-\frac{\kappa\omega}{\sqrt{\lambda^2+\kappa^2}}t\right)+\kappa\sin\left(2\lambda\gamma+\frac{\lambda\omega}{\sqrt{\lambda^2+\kappa^2}}x-\frac{\lambda\sqrt{\omega^2+1}}{\sqrt{\lambda^2+\kappa^2}}t\right)}{\lambda\sinh\left(2\kappa\beta+\frac{\kappa\sqrt{\omega^2+1}}{\sqrt{\lambda^2+\kappa^2}}x-\frac{\kappa\omega}{\sqrt{\lambda^2+\kappa^2}}t\right)-\kappa\sin\left(2\lambda\gamma+\frac{\lambda\omega}{\sqrt{\lambda^2+\kappa^2}}x-\frac{\lambda\sqrt{\omega^2+1}}{\sqrt{\lambda^2+\kappa^2}}t\right)} \pmod{2\pi}, \tag{1.9}$$

where $\omega = \frac{4\kappa^2+4\lambda^2-1}{4\sqrt{\kappa^2+\lambda^2}}$, called correspondingly *imaginary breathers* (if the upper signs are taken) and *imaginary antibreathers* (if the lower signs are taken), are obtained by taking $\varphi = \frac{\pi}{2}$ in (1.7). A snapshot of an imaginary breather is shown in figure 5.

Solutions (1.7), including special cases (1.8), (1.9), are superpositions of two complex-valued solitons (1.2) generated by two charges c_1, c_2 in the μ-plane. By analogy with electrostatic theory we may consider the limiting case of (1.7) as the distance 2λ between the charges in the μ-space goes to zero while the value of the charges $c_1, c_2 = -c_1$ increases at the order of $\frac{1}{\lambda}$. The limiting case gives us

$$f = \mp 2\mathrm{i}\ln\frac{\cosh\left[\kappa\left(2\beta+2\xi+\frac{1}{2\kappa^2}\eta\right)+\mathrm{i}\varphi\right]+\mathrm{i}\kappa\left(2\gamma+2\xi-\frac{1}{2\kappa^2}\eta\right)}{\cosh\left[\kappa\left(2\beta+2\xi+\frac{1}{2\kappa^2}\eta\right)+\mathrm{i}\varphi\right]-\mathrm{i}\kappa\left(2\gamma+2\xi-\frac{1}{2\kappa^2}\eta\right)} \pmod{2\pi}, \tag{1.10}$$

which, again by analogy with electrostatic theory, we call *solitonic dipoles*. A more detailed description of solitonic dipoles is given in [Wu1] albeit under a different name.

Another limiting case of (1.7)

$$f = \mp 2\mathrm{i}\ln\frac{\lambda\left(2\beta+2\xi+\frac{1}{2\lambda^2}\eta\right)+\sin\lambda\left(2\gamma+2\xi-\frac{1}{2\lambda^2}\eta\right)}{\lambda\left(2\beta+2\xi+\frac{1}{2\lambda^2}\eta\right)-\sin\lambda\left(2\gamma+2\xi-\frac{1}{2\lambda^2}\eta\right)} \pmod{2\pi}$$

$$= \mp 2\mathrm{i}\ln\frac{(2\lambda\beta+\sqrt{\omega^2+1}\,x-\omega t)+\sin(2\lambda\gamma+\omega x-\sqrt{\omega^2+1}\,t)}{(2\lambda\beta+\sqrt{\omega^2+1}\,x-\omega t)-\sin(2\lambda\gamma+\omega x-\sqrt{\omega^2+1}\,t)} \pmod{2\pi}, \tag{1.11}$$

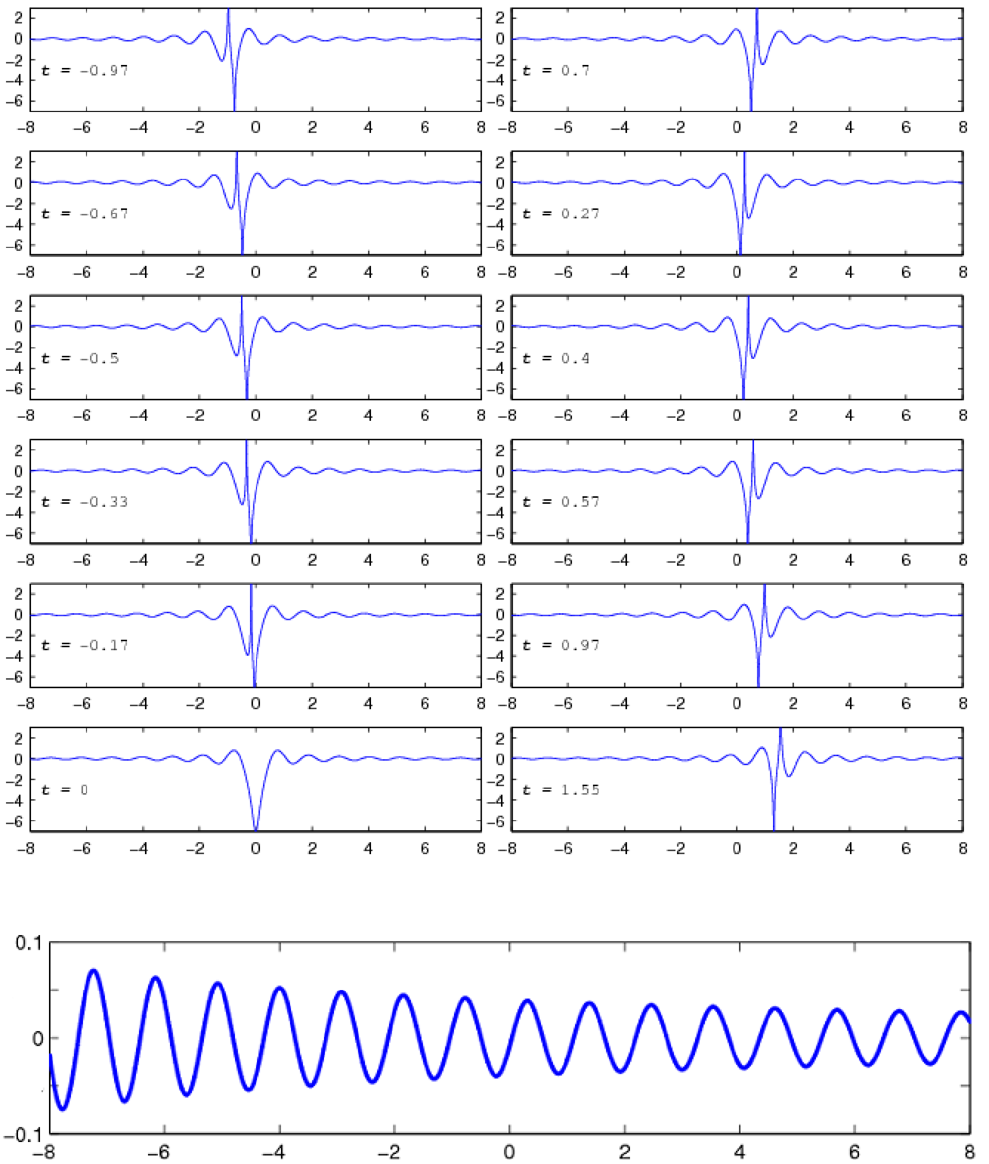

Figure 6. Time evolution of the negative of the imaginary part $(\mathrm{i}f)$ of a harmonic breather with $\lambda = 3,\ \beta = \gamma = 0$ and a snapshot of the negative of the imaginary part $(\mathrm{i}f)$ of the oscillating tail of a harmonic breather with $\lambda = 3,\ \beta = 15,\ \gamma = 0$.

where $\omega = \frac{4\lambda^2-1}{4\lambda}$, is obtained by taking $\varphi = \frac{\pi}{2}$ (which actually gives us (1.9)) and letting $\kappa \to 0$. We shall call (1.11) *harmonic breathers*; they were first introduced in [Beu1] and further studied in [And1] albeit under a different name.

Breathers (1.7) as well as their limiting cases (1.10) vary in shape depending on the values of κ and λ but all exhibit exponential decay as $x \to \pm\infty$. Time of evolution of the imaginary part of a typical harmonic breather is shown in figure 6.

Harmonic breathers, however, decay much slower as $x \to \pm\infty$; the rate of decay at $\pm\infty$ provides harmonic breathers with rather interesting properties that we shall study in this paper.

Note that the frequencies of breathers (1.7) as well as the limiting cases (1.10) (1.11) may be changed by a Lorentz transformation

$$\begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{1-v^2}} & -\frac{v}{\sqrt{1-v^2}} \\ -\frac{v}{\sqrt{1-v^2}} & \frac{1}{\sqrt{1-v^2}} \end{pmatrix} \begin{pmatrix} t' \\ x' \end{pmatrix} \tag{1.12a}$$

$$\begin{pmatrix} \omega \\ \sqrt{1+\omega^2} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{1-v^2}} & -\frac{v}{\sqrt{1-v^2}} \\ -\frac{v}{\sqrt{1-v^2}} & \frac{1}{\sqrt{1-v^2}} \end{pmatrix} \begin{pmatrix} \omega' \\ \sqrt{1+(\omega')^2} \end{pmatrix}. \tag{1.12b}$$

2. Superposition formula for solitons and harmonic breathers: properties of multi-harmonic breather solutions

Just like superposition of M solitons is described by (1.1) superposition of N harmonic breathers and M kinks/antikinks is described by

$$f(x,t) = -2\mathrm{i}\ln\frac{\det K_+}{\det K_-} \quad (\mathrm{mod}\, 2\pi), \tag{2.1a}$$

where $K_\pm$ is an $(N+M)\times(N+M)$ symmetric matrix with the entries

$$K_{\pm\, nm} = \begin{cases} \delta_{mn}B_n + (1-\delta_{mn})\dfrac{\sin(\Gamma_n-\Gamma_m)}{\lambda_n-\lambda_m} \pm \dfrac{\sin(\Gamma_n+\Gamma_m)}{\lambda_n+\lambda_m}, \qquad \text{for } n,m=1,2,\ldots,N, \\ \dfrac{\sqrt{\mu_n}\,\mathrm{e}^{A_n}}{\mu_n^2+\lambda_m^2}\left[\dfrac{1\mp 1}{2}(\lambda_m\cos\Gamma_m - \mu_n\sin\Gamma_m) + \dfrac{1\pm 1}{2}(\mu_n\cos\Gamma_m+\lambda_m\sin\Gamma_m)\right] \\ \text{for } n=N+1,N+2,\ldots,N+M, \quad m=1,2,\ldots,N, \\ \text{the same as the previous line with } n \text{ and } m \text{ interchanged}, \\ \text{for } n=1,2,\ldots,N, \quad m=N+1,N+2,\ldots,N+M, \\ \pm\dfrac{1}{4}\delta_{nm} + \dfrac{\sqrt{\mu_n\mu_m}}{2(\mu_n+\mu_m)}\,\mathrm{e}^{A_n+A_m} \text{ for } n,m=2N+1,2N+2,\ldots,2N+M, \end{cases} \tag{2.1b}$$

and

$$\begin{aligned} A_n &= \mu_n\alpha_n + \frac{\varphi_n}{2}\mathrm{i} + \mu_n\xi + \frac{1}{4\mu_n}\eta = \mu_n\alpha_n + \frac{\varphi_n}{2}\mathrm{i} + \frac{4\mu_n^2+1}{8\mu_n}x - \frac{4\mu_n^2-1}{8\mu_n}t, \\ B_n &= \beta_n + \xi + \frac{1}{4\lambda_n^2}\eta = \beta_n + \frac{4\lambda_n^2+1}{8\lambda_n^2}x - \frac{4\lambda_n^2-1}{8\lambda_n^2}t, \\ \Gamma_n &= \lambda_n\gamma_n + \lambda_n\xi - \frac{1}{4\lambda_n}\eta = \lambda_n\gamma_n + \frac{4\lambda_n^2-1}{8\lambda_n}x - \frac{4\lambda_n^2+1}{8\lambda_n}t, \\ &\lambda_n > 0, \quad \beta_n \in \mathbb{R}, \quad \gamma_n \in \mathbb{R}, \quad \mu_n > 0, \quad \alpha_n \in \mathbb{R}, \quad \varphi_n \in [0, 2\pi), \\ \eta &= \frac{x+t}{2}, \qquad \xi = \frac{x-t}{2}. \end{aligned} \tag{2.1c}$$

The jth diagonal element describes a real kink, a real antikink, an imaginary kink or an imaginary antikink if the value of φ_j is correspondingly $\frac{\pi}{2}, -\frac{\pi}{2}, 0, \pi$.

The superposition of M kinks/antikinks, $\widetilde{M}$ weak kinks/antikinks and N harmonic breathers is given by

$$f(x,t) = -2\mathrm{i}\ln\frac{\det K_+}{\det K_-} + f_{w1}(x-t) + f_{w2}(x+t) \quad (\mathrm{mod}\, 2\pi), \tag{2.2}$$

where $K_\pm$ are the same as in (2.1) and $f_{w1}(x-t)$, $f_{w2}(x+t)$ are two step functions attaining values from the set $0, \pm 2\pi, \pm 4\pi, \ldots, \pm 2M\pi$ of multiples of 2π with the first function $f_{w1}(x-t)$ being a function of $x-t$ only while the second function $f_{w2}(x+t)$ being a function of $x+t$ only. The derivation of (2.1) and (2.2) is given in appendix A.

As a special case of (2.1), we obtain that the superposition of N harmonic breathers with no solitons present is given by

$$f(x,t) = -2\mathrm{i}\ln\frac{\det \mathcal{K}_+}{\det \mathcal{K}_-} \quad (\mathrm{mod}\, 2\pi), \tag{2.3a}$$

where $\mathcal{K}_\pm$ is an $N\times N$ symmetric matrix with entries

$$\mathcal{K}_{\pm\, nm} = \delta_{mn}B_n + (1-\delta_{mn})\frac{\sin(\Gamma_n-\Gamma_m)}{\lambda_n-\lambda_m} \pm \frac{\sin(\Gamma_n+\Gamma_m)}{\lambda_n+\lambda_m}, \tag{2.3b}$$

and

$$\begin{aligned} B_n &= \beta_n + \xi + \frac{1}{4\lambda_n^2}\eta = \frac{2\lambda_n\beta_n + \sqrt{\omega_n^2+1}\,x - \omega_n\,t}{2\lambda_n}, \\ \Gamma_n &= \lambda_n\gamma_n + \lambda_n\xi - \frac{1}{4\lambda_n}\eta = \frac{2\lambda_n\gamma_n + \omega_n x - \sqrt{\omega_n^2+1}\,t}{2}, \\ \lambda_n &> 0, \quad \beta_n\in\mathbb{R}, \quad \gamma_n\in\mathbb{R}, \quad \eta = \frac{x+t}{2}, \quad \xi = \frac{x-t}{2}, \quad \omega_n = \frac{4\lambda_n^2-1}{4\lambda_n}. \end{aligned} \tag{2.3c}$$

Each solution (2.3) has $2N$ singularities for each value of t; the proof is given in appendix B. Away from the singularities, the entries of $\mathcal{K}_\pm$ in (2.3) are dominated by $B_n = \beta_n + \xi + \frac{1}{4\lambda_n^2}\eta$ and $\det\mathcal{K}_\pm \approx \prod_{n=1}^N\left(\beta_n + \xi + \frac{1}{4\lambda_n^2}\eta \pm \frac{\sin 2\Gamma_n}{2\lambda_n}\right)$; hence

$$-2\mathrm{i}\ln\frac{\det\mathcal{K}_+}{\det\mathcal{K}_-} \approx -2\mathrm{i}\ln\prod_{n=1}^N \frac{\beta_n+\xi+\frac{1}{4\lambda_n^2}\eta + \frac{\sin 2\Gamma_n}{2\lambda_n}}{\beta_n+\xi+\frac{1}{4\lambda_n^2}\eta - \frac{\sin 2\Gamma_n}{2\lambda_n}} \approx -2\mathrm{i}\sum_{n=1}^N \frac{\sin 2\Gamma_n}{\lambda_n\beta_n + \lambda_n\xi + \frac{1}{4\lambda_n}\eta}.$$

Thus, sufficiently far away from their singularities functions (2.3) satisfy

$$\begin{aligned} f(x,t) &\approx -2\mathrm{i}\sum_{n=1}^N \frac{\sin 2\Gamma_n}{\lambda_n\beta_n + \lambda_n\xi + \frac{1}{4\lambda_n}\eta} \quad (\mathrm{mod}\, 2\pi) \\ &= -4\mathrm{i}\sum_{n=1}^N \frac{\sin\left(2\lambda_n\gamma_n + \omega_n\,x - \sqrt{\omega_n^2+1}\,t\right)}{2\lambda_n\beta_n + \sqrt{\omega_n^2+1}\,x - \omega_n\,t} \quad (\mathrm{mod}\, 2\pi). \end{aligned} \tag{2.4a}$$

If $|x|$, $|t|$ are restricted to a region whose size is much smaller than all $|\beta_n|$, then estimate (2.4a) may be further simplified to

$$f(x,t) \approx -2\mathrm{i}\sum_{n=1}^N \frac{1}{\lambda_n\beta_n}\sin\left(2\lambda_n\gamma_n + \omega_n\,x - \sqrt{\omega_n^2+1}\,t\right) \quad (\mathrm{mod}\, 2\pi). \tag{2.4b}$$

Functions (2.3) also exhibit the phenomenon of *nonlinear interference* very similar to its linear namesake. Specifically, if $\mathcal{K}_\pm$ is given by (2.3b), $\gamma_m = \gamma(\lambda_m)$ where $\gamma(\lambda)$ is a C^1 function of λ, then as $\lambda_m \to \lambda_{m-1}$,

$$\det\mathcal{K}_\pm \to \det\widetilde{\mathcal{K}}_\pm, \tag{2.5a}$$

where $\widetilde{\mathcal{K}_\pm}$ is obtained from $\mathcal{K}_\pm$ by removing the mth row and mth column, and replacing β_{m-1} with

$$\widetilde{\beta}_{m-1} = \frac{\beta_m\ \beta_{m-1} - \left[\frac{\mathrm{d}\left(\lambda_m\gamma(\lambda_m)\right)}{\mathrm{d}\lambda_m}\right]^2}{\beta_m + \beta_{m-1} - 2\,\frac{\mathrm{d}\left(\lambda_m\gamma(\lambda_m)\right)}{\mathrm{d}\lambda_m}},$$

the latter is equivalent to

$$\frac{1}{\widetilde{\beta_m} - \frac{\mathrm{d}(\lambda_m\gamma(\lambda_m))}{\mathrm{d}\lambda_m}} = \frac{1}{\beta_m - \frac{\mathrm{d}(\lambda_m\gamma(\lambda_m))}{\mathrm{d}\lambda_m}} + \frac{1}{\beta_{m-1} - \frac{\mathrm{d}(\lambda_{m-1}\gamma(\lambda_{m-1}))}{\mathrm{d}\lambda_{m-1}}}. \tag{2.5b}$$

Formula (2.5) shows that for a given function $\gamma(\lambda)$, the superposition of two harmonic breathers with $\lambda_1 = \lambda_2,\quad \gamma_1 = \gamma_2$ and $\beta_1 + \beta_2 - 2\frac{\mathrm{d}(\lambda\gamma(\lambda))}{\mathrm{d}\lambda}|_{\lambda=\lambda_1} = 0$ is $0(\mathrm{mod}2\pi)$ suggesting that such breathers may be viewed as annihilators of each other, the same way as solitons and antisolitons, and hence represent a harmonic breather and its harmonic antibreather; which one of the two is harmonic breather and which is harmonic antibreather is a matter of choice.

Formulas (2.5) also suggest that asymptotics (2.4) be replaced correspondingly with

$$f(x,t) \approx -4\mathrm{i}\sum_{n=1}^{N} \frac{\sin\left(2\lambda_n\gamma_n + \omega_n\ x - \sqrt{\omega_n^2+1}\ t\right)}{2\lambda_n\beta_n - 2\lambda_n\frac{\mathrm{d}(\lambda_n\gamma(\lambda_n))}{\mathrm{d}\lambda_n} + \sqrt{\omega_n^2+1}\ x - \omega_n\ t} \quad (\mathrm{mod}\ 2\pi). \tag{2.6a}$$

$$f(x,t) \approx -2\mathrm{i}\sum_{n=1}^{N} \frac{1}{\lambda_n\beta_n - \lambda_n\frac{\mathrm{d}(\lambda_n\gamma(\lambda_n))}{\mathrm{d}\lambda_n}} \sin(2\lambda_n\gamma_n + \omega_n\ x - \sqrt{\omega_n^2+1}\ t) \quad (\mathrm{mod}\ 2\pi), \tag{2.6b}$$

provided $\sup_{\lambda\geqslant 0}\left|\frac{\mathrm{d}(\lambda\gamma(\lambda))}{\mathrm{d}\lambda}\right|$ is much smaller than all $|\beta_n|$. Formula (2.6*b*) provides a Fourier series approximation to the solutions (2.3) away from the singularities. Asymptotics (2.6) suggest the use of harmonic breathers as building blocks to construct more complicated solutions of the sine-Gordon equation similarly to how the trigonometric functions are used to construct more complicated functions by means of the Fourier transform. Such construction may use either the left or right oscillating tails of harmonic breathers to modulate functions on a finite space-time domain $|x| < X,\quad |t| < T$, as long as the singularities are sufficiently far away from the domain.

Since both left and right tails may be used for modulations, one may try to use both. One way to do it would be to consider superpositions of an even number $2N$ of harmonic breathers with

$$\begin{aligned}
&\lambda_{2n-1} - \lambda_{2n} \text{ very small}, \qquad \gamma(\lambda) \in C^1([0,+\infty)),\\
&\lambda_{2n-1}\gamma_{2n-1} = \lambda_{2n-1}\gamma(\lambda_{2n-1}), \qquad \lambda_{2n}\gamma_{2n} = \lambda_{2n}\gamma(\lambda_{2n}) + \frac{\pi}{2},\\
&\beta_{2n-1} = \frac{\rho(\lambda_{2n-1})}{\lambda_{2n-1}-\lambda_{2n}} + \frac{\mathrm{d}(\lambda\gamma(\lambda))}{\mathrm{d}\lambda}, \qquad \beta_{2n} = -\frac{\rho(\lambda_{2n})}{\lambda_{2n-1}-\lambda_{2n}} + \frac{\mathrm{d}(\lambda\gamma(\lambda))}{\mathrm{d}\lambda},\\
&\max_{\lambda\geqslant 0}\left|\frac{\mathrm{d}\left(\lambda\gamma(\lambda)\right)}{\mathrm{d}\lambda}\right| \ll \inf_{\lambda\geqslant 0}\rho(\lambda),
\end{aligned} \tag{2.7a}$$

where

$$\rho(\lambda) \in C^1([0,+\infty)), \qquad \rho(\lambda) > 0. \tag{2.7b}$$

We shall call a superposition of $2N$ harmonic breathers subject to (2.7) a *double layer*. If $\min_{\lambda\geqslant 0}\rho(\lambda) > 0$ and $|\lambda_1 - \lambda_2|$ is sufficiently small, then the corresponding double layer has a region close to the origin; we shall call it *core*, where modulation is also possible. In the core (or any other region sufficiently far away from the singularities where $\frac{\rho(\lambda_1)}{|\lambda_1-\lambda_2|}$, $\frac{1}{|\lambda_1-\lambda_2|}$ are much larger than all other terms), we may estimate a double layer composed of only two breathers as

$$
\begin{aligned}
f(x,t) &= -2\mathrm{i}\ln\frac{\det\begin{vmatrix} B_1+\frac{\sin 2\Gamma_1}{2\lambda_1} & \frac{\sin(\Gamma_1-\Gamma_2)}{\lambda_1-\lambda_2}+\frac{\sin(\Gamma_1+\Gamma_2)}{\lambda_1+\lambda_2} \\ \frac{\sin(\Gamma_1-\Gamma_2)}{\lambda_1-\lambda_2}+\frac{\sin(\Gamma_1+\Gamma_2)}{\lambda_1+\lambda_2} & B_2+\frac{\sin 2\Gamma_2}{2\lambda_2}\end{vmatrix}}{\det\begin{vmatrix} B_1-\frac{\sin 2\Gamma_1}{2\lambda_1} & \frac{\sin(\Gamma_1-\Gamma_2)}{\lambda_1-\lambda_2}-\frac{\sin(\Gamma_1+\Gamma_2)}{\lambda_1+\lambda_2} \\ \frac{\sin(\Gamma_1-\Gamma_2)}{\lambda_1-\lambda_2}-\frac{\sin(\Gamma_1+\Gamma_2)}{\lambda_1+\lambda_2} & B_2-\frac{\sin 2\Gamma_2}{2\lambda_2}\end{vmatrix}} \\
&\approx -2\mathrm{i}\ln\frac{\det\begin{vmatrix} B_1+\frac{\sin 2\Gamma_1}{2\lambda_1} & -\frac{1}{\lambda_1-\lambda_2}+\frac{\cos 2\Gamma_1}{2\lambda_1} \\ -\frac{1}{\lambda_1-\lambda_2}+\frac{\cos 2\Gamma_1}{2\lambda_1} & B_2-\frac{\sin 2\Gamma_1}{2\lambda_1}\end{vmatrix}}{\det\begin{vmatrix} B_1-\frac{\sin 2\Gamma_1}{2\lambda_1} & -\frac{1}{\lambda_1-\lambda_2}-\frac{\cos 2\Gamma_1}{2\lambda_1} \\ -\frac{1}{\lambda_1-\lambda_2}-\frac{\cos 2\Gamma_1}{2\lambda_1} & B_2+\frac{\sin 2\Gamma_1}{2\lambda_1}\end{vmatrix}} \\
&\approx -2\mathrm{i}\ln\frac{B_1B_2-(B_1-B_2)\frac{\sin 2\Gamma_1}{2\lambda_1}-\frac{1}{(\lambda_1-\lambda_2)^2}+\frac{\cos 2\Gamma_1}{\lambda_1(\lambda_1-\lambda_2)}}{B_1B_2+(B_1-B_2)\frac{\sin 2\Gamma_1}{2\lambda_1}-\frac{1}{(\lambda_1-\lambda_2)^2}-\frac{\cos 2\Gamma_1}{\lambda_1(\lambda_1-\lambda_2)}} \\
&\approx -2\mathrm{i}\ln\frac{\lambda_1[\rho(\lambda_1)^2+1]+(\lambda_1-\lambda_2)[\rho(\lambda_1)\sin 2\Gamma_1-\cos 2\Gamma_1]}{\lambda_1[\rho(\lambda_1)^2+1]-(\lambda_1-\lambda_2)[\rho(\lambda_1)\sin 2\Gamma_1-\cos 2\Gamma_1]} \\
&= -2\mathrm{i}\ln\frac{\lambda_1\sqrt{\rho(\lambda_1)^2+1}+(\lambda_1-\lambda_2)\sin\left(2\Gamma_1-\arctan\frac{1}{\rho(\lambda_1)}\right)}{\lambda_1\sqrt{\rho(\lambda_1)^2+1}-(\lambda_1-\lambda_2)\sin\left(2\Gamma_1-\arctan\frac{1}{\rho(\lambda_1)}\right)} \\
&\approx -4\mathrm{i}\frac{(\lambda_1-\lambda_2)\sin\left(2\Gamma_1-\arctan\frac{1}{\rho(\lambda_1)}\right)}{\lambda_1\sqrt{\rho(\lambda_1)^2+1}}.
\end{aligned}
\tag{2.8a}
$$

The evolution of a harmonic couple is shown in figure 7. The shape of the core is practically indistinguishable from that of the corresponding trigonometric function.

If $\rho(\lambda_1) \gg 1$, the asymptotic (2.8) coincides with the asymptotic obtained from (2.6). Asymptotic (2.8) suggests that a more general double layer solution of the sine-Gordon equation satisfies

$$
f(x,t) \approx -4\mathrm{i}\sum_{n=1}^{N}\frac{(\lambda_{2n-1}-\lambda_{2n})\sin\left(2\Gamma_{2n-1}-\arctan\frac{1}{\rho(\lambda_{2n-1})}\right)}{\lambda_{2n-1}\sqrt{\rho(\lambda_{2n-1})^2+1}}. \tag{2.8b}
$$

3. Examples of modulation by harmonic breathers/couples

In this section, we provide examples of modulation by the tails of harmonic breathers and the cores of harmonic couples using correspondingly asymptotics (2.6) and (2.8). Note that outside the intervals of modulation shown on the graphs below the solutions eventually develop singularities.

In the first example shown in figure 8, we construct a solution of the sine-Gordon equation whose initial profile is very similar to a multiple of the Dirac δ-function. Note that there can be no solution of the sine-Gordon equation whose initial profile is exactly a multiple of the δ-function for the construction of such a solution would require $\sin[C\delta(x)]$ to be well defined at least for some nonzero constant C which, of course, is not the case. Since away from the singularities solutions (2.3) asymptotically behave like (2.4b), the sought solution of the sine-Gordon equation must be asymptotically proportional to the Fourier series

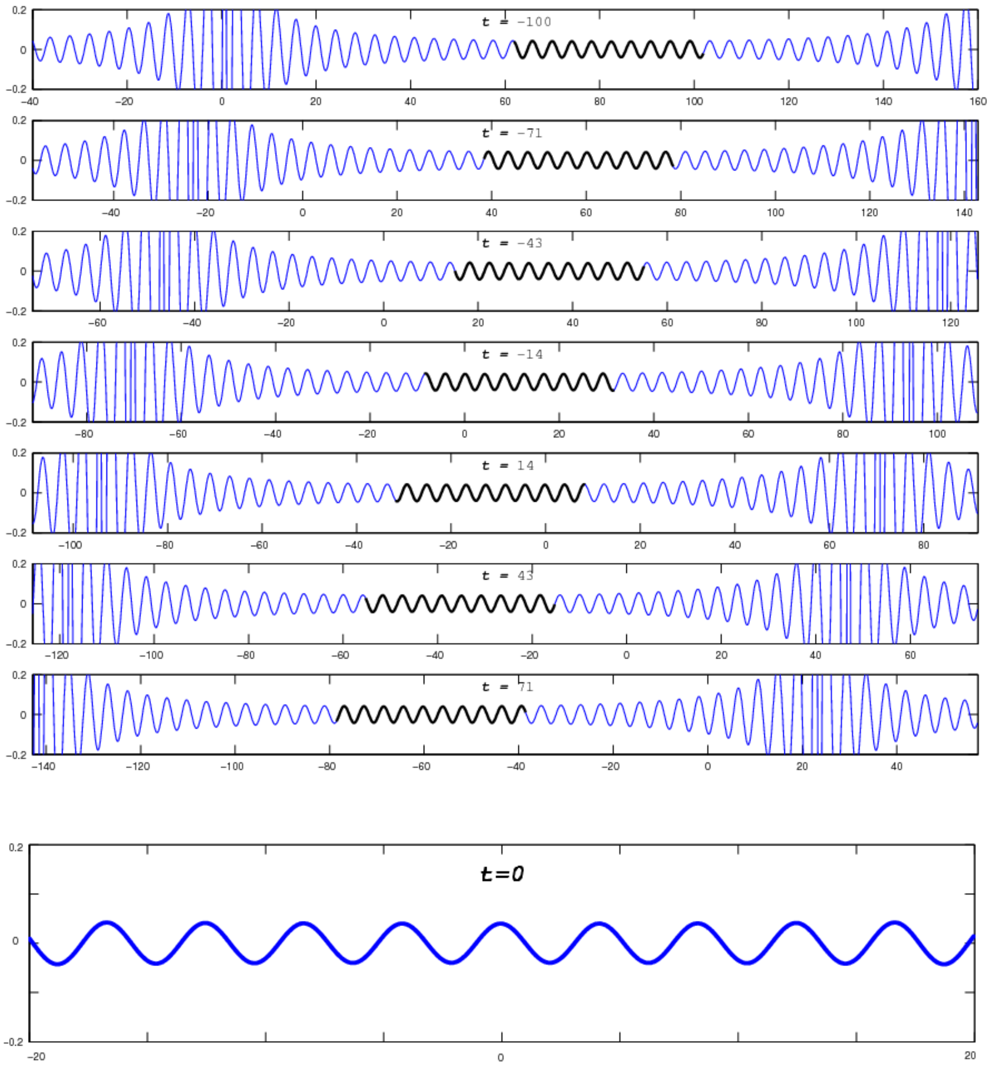

Figure 7. Time evolution and a snapshot of negative of the imaginary part (if) of the oscillatory core of a double layer composed of only two harmonic breathers with $\lambda_1 = -0.15$, $\lambda_2 = -0.1515$, $\beta_1 = 50$, $\beta_2 = -50$, $\gamma_1 = 0$, $\gamma_2 = \frac{\pi}{2}$.

$\frac{2}{\pi}\lim_{\Delta\lambda\to 0}\left[\Delta\lambda\sum_{n=0}^{+\infty}\cos 2\lambda_n x\right]$ approximating Fourier integral $\frac{1}{\pi}\int_{-\infty}^{+\infty}\cos 2\lambda x\,\mathrm{d}\lambda = \delta(x)$. Motivated by this, we define $\Delta\omega = 0.12$, $\omega_{2n-0.5} = (2n-0.5)\Delta\omega$, $\omega_{2n} = 2n\Delta\omega$, $\omega_{2n-1} = (2n-1)\Delta\omega$ and choose the parameters in (2.3) to be

$$\lambda_{2n-1} = \frac{\sqrt{\omega_{2n-0.5}^2+1}-\omega_{2n-0.5}}{2} - 0.2\left[\frac{\sqrt{\omega_{2n}^2+1}-\omega_{2n}}{2} - \frac{\sqrt{\omega_{2n-1}^2+1}-\omega_{2n-1}}{2}\right],$$

$$\lambda_{2n-1} = \frac{\sqrt{\omega_{2n-0.5}^2+1}-\omega_{2n-0.5}}{2} + 0.2\left[\frac{\sqrt{\omega_{2n}^2+1}-\omega_{2n}}{2} - \frac{\sqrt{\omega_{2n-1}^2+1}-\omega_{2n-1}}{2}\right],$$

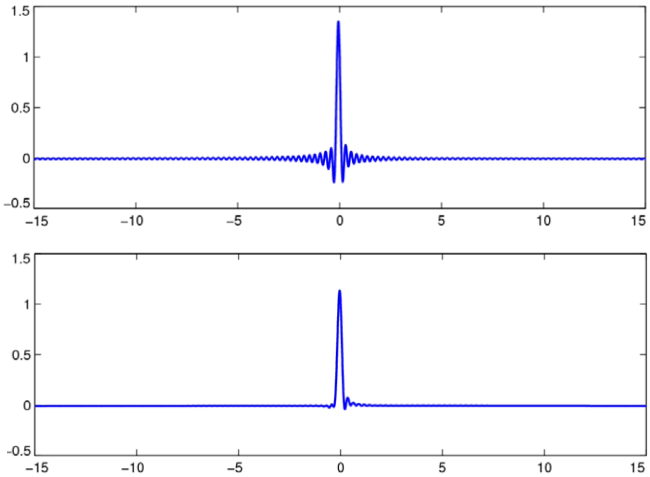

Figure 8. An approximation to the initial profile similar to the $\delta(x)$-function.

$\gamma_{2n-1} = \frac{\pi}{4\lambda_{2n-1}}$, $\gamma_{2n} = \frac{3\pi}{4\lambda_{2n}}$, $\beta_{2n-1} = \frac{105}{\lambda_{2n-1}}$, $\beta_{2n} = -\frac{300}{\lambda_{2n}}$with $1 \leqslant n \leqslant N = 100$. The graph of the negative of the imaginary part of the so-obtained solution of the sine-Gordon equation at $t = 0$ is shown in the top portion of figure 8. The little horns pointing down both on the left and right of the big spike in the middle is the nonlinear analog of the Gibbs phenomenon; they are eliminated by replacing formulas for β_{2n-1}, β_{2n} with $\beta_{2n-1} = \frac{105}{\lambda_{2n-1}} \cdot \frac{2N}{2N-2n+0.5}$, $\beta_{2n} = -\frac{300}{\lambda_{2n}} \cdot \frac{2N}{2N-2n+0.5}$. The obtained graph is shown in the bottom portion of figure 8.

Similarly we may construct the solutions of the sine-Gordon equation whose initial profile is similar to $f_0(x) = \begin{cases} 1, & \text{if } |x|>2, \\ 0, & \text{if } |x|<2. \end{cases}$ Since away from the singularities solutions (2.3) asymptotically behave like (2.4*b*), the sought solution of the sine-Gordon equation must be asymptotically proportional to the Fourier series $\frac{2}{\pi} \lim_{\Delta\lambda \to 0} \sum_{n=0}^{+\infty} \frac{\Delta\lambda \, \sin 4\lambda_n \, \cos 2\lambda_n x}{\lambda_n}$ approximating Fourier integral $\frac{2}{\pi} \int_{-\infty}^{+\infty} \frac{\sin 4\lambda \, \cos 2\lambda x}{\lambda} \, \mathrm{d}\lambda = f_0(x)$. Motivated by this, we define $\Delta\omega = 0.02$, $\omega_{2n-0.5} = (2n - 0.5)\Delta\omega$, $\omega_{2n} = 2n\Delta\omega$, $\omega_{2n-1} = (2n - 1)\Delta\omega$ and choose the parameters in (2.3) to be

$$\lambda_{2n-1} = \frac{\sqrt{\omega_{2n-0.5}^2 + 1} + \omega_{2n-0.5}}{2} - 0.2 \left[\frac{\sqrt{\omega_{2n}^2 + 1} + \omega_{2n}}{2} - \frac{\sqrt{\omega_{2n-1}^2 + 1} + \omega_{2n-1}}{2} \right],$$

$$\lambda_{2n-1} = \frac{\sqrt{\omega_{2n-0.5}^2 + 1} + \omega_{2n-0.5}}{2} + 0.2 \left[\frac{\sqrt{\omega_{2n}^2 + 1} + \omega_{2n}}{2} - \frac{\sqrt{\omega_{2n-1}^2 + 1} + \omega_{2n-1}}{2} \right],$$

$\gamma_{2n-1} = \frac{\pi}{4\lambda_{2n-1}}$, $\gamma_{2n} = \frac{3\pi}{4\lambda_{2n}}$, $\beta_{2n-1} = \frac{15(2n-1)}{\lambda_{2n-1} \sin[2(2n-1)\Delta\omega]}$, $\beta_{2n} = -\frac{30n}{\lambda_{2n} \sin[4n\Delta\omega]}$ with $1 \leqslant n \leqslant N = 400$. The negative of the imaginary part of the graph of the so-obtained solution of the sine-Gordon equation at $t = 0$ is shown in the first two pictures of figure 9; the second picture is just a more detailed version of the first one. The spikes pointing up and down at $x = \pm 2$ is the Gibbs phenomenon; they are eliminated by replacing formulas for β_{2n-1}, β_{2n} with $\beta_{2n-1} = \frac{15(2n-1)}{\lambda_{2n-1} \sin[2(2n-1)\Delta\omega]} \cdot \frac{2N}{2N-2n+0.5}$, $\beta_{2n} = -\frac{30n}{\lambda_{2n} \sin[4n\Delta\omega]} \cdot \frac{2N}{2N-2n+0.5}$. The obtained graph is shown in the third picture of figure 9.

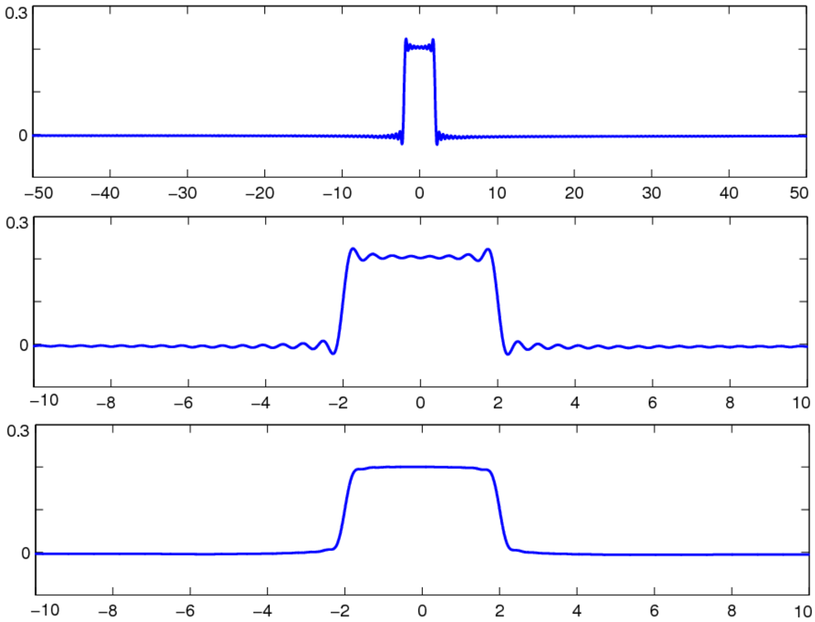

Figure 9. Approximations to the step function $f_0(x) = \begin{cases} 1 & \text{if } |x|>2 \\ 0 & \text{if } |x|<2 \end{cases}$.

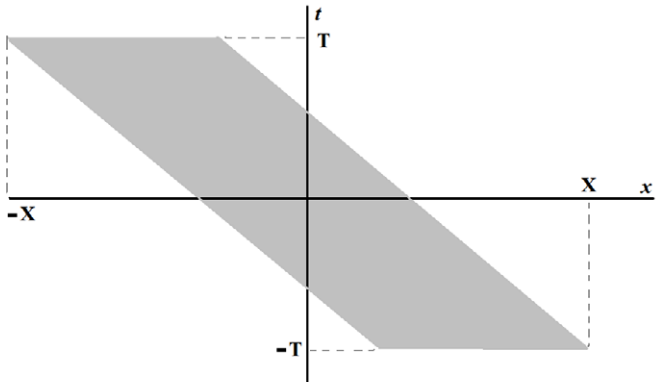

Figure 10. Sketch of a typical space–time domain of modulation.

As already mentioned in the beginning of this section, as we proceed to either left or right from the origin the solutions shown in figures 8 and 9 eventually develop singularities. However, the interval where the modulation is valid is sufficiently large; since in applications intervals are never of infinite length, a sufficiently large interval of modulation is practically as good as $\mathbb{R}$. In space–time, the modulation is valid in a strip of the form shown in figure 10; the exact values of X and T depend on the functions modulated.

Although the profiles shown in figures 8 and 9 have extremely short lifespan, their decay is fairly slow. Figure 11 shows time evolution of the profile shown in figure 8, as can be seen the profile decays into a wave packet that still exists at time $t = \pm 500$. Figure 12 shows time

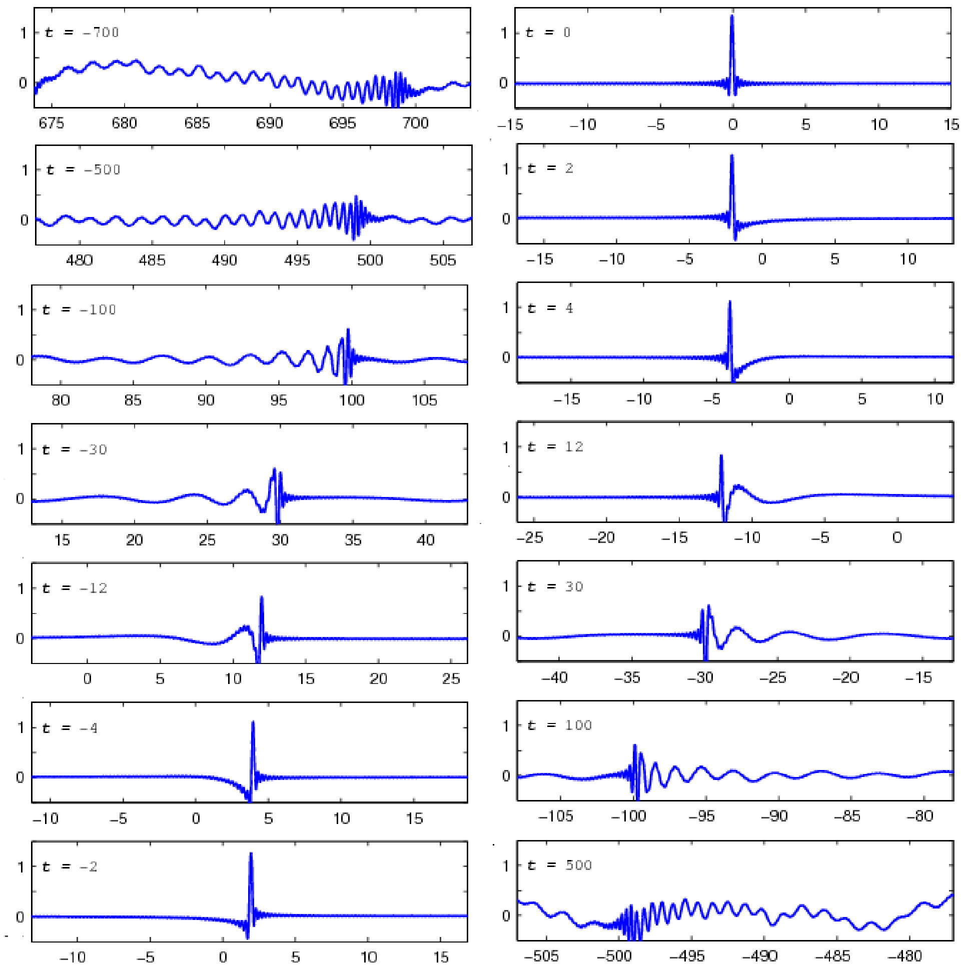

Figure 11. Time evolution of the profile shown in figure 8.

evolution of the profile shown in figure 9, as can be seen the profile decays much faster than the previous one.

One may construct solutions of the sine-Gordon equation with a relatively long lifespan. An example of one such solution, obtained by taking $N = 100$, $\Delta\omega = 0.08$, $\omega_0 = 8$, $\omega_n = \omega_0 + \Delta\omega(n - 50.5)$, $\lambda_n = \frac{\omega_n + \sqrt{\omega_n^2+1}}{2}$, $\beta_0 = 168$, $\beta_n = (-1)^n \frac{\beta_0}{\lambda_n} \cdot \mathrm{e}^{0.06(\omega_n-\omega_0)^2}$, $\gamma_{2n-1} = 0$, $\gamma_{2n} = \frac{\pi}{2}$, is shown in figure 12. The wave packet preserves its shape for $|t| < 30$, shows significant changes only at $t \sim \pm 100$ and does not completely disperse even at $t \sim 1500$. Reducing the values of λ, we may extend the lifespan of the wave packet even further.

4. Sine-Gordon universes

Since the sine-Gordon equation is odd in f, if a function $f(x, t)$ satisfies the sine-Gordon equation so does $-f(x, t)$. A singular non-zero solution of the sine-

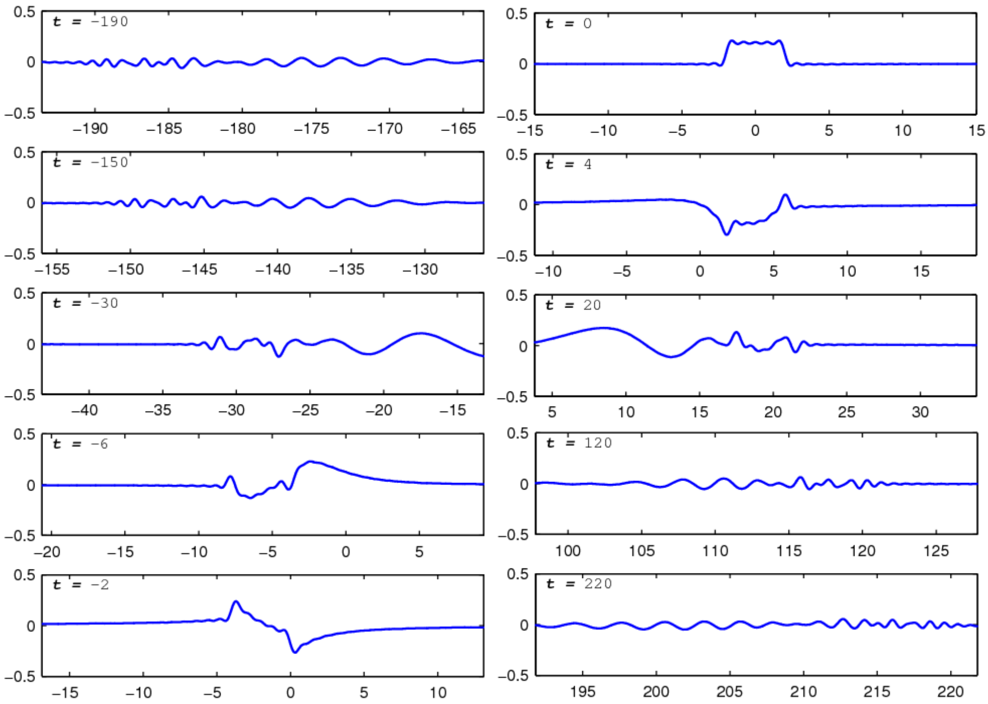

Figure 12. Time evolution of the profile shown in figure 9.

Gordon equation $f(x,\ t)$ with S singularities at $x_{s1}(t), x_{s2}(t), x_{s3}(t), \ldots, x_{sS}(t)$, $x_{s1}(t) \leqslant x_{s2}(t) \leqslant x_{s3}(t) \leqslant \cdots \leqslant x_{sS}(t)$ is regular on each of the intervals $(-\infty, x_{s1}(t))$, $(x_{s1}(t), x_{s2}(t)), (x_{s2}(t), x_{s3}(t)), \ldots, (x_{sS-1}(t), x_{sS}(t)), (x_{sS}(t), +\infty)$. Multiplying function $f(x,\ t)$ by -1 or 0 on one of the intervals produces another solution of the sine-Gordon equation; the procedure allows us to generate 3^{S+1} distinct solutions (including the zero solution). Thus, for any two consecutive singularities $x_{sk}(t)$, $x_{sk+1}(t)$, of $f(x,t)$, we may construct a solution

$$\tilde{f}(x,t) = \begin{cases} f(x,t), & \text{if} \quad x_{sk}(t) < x < x_{sk+1}(t), \\ 0, & \text{otherwise.} \end{cases}$$

Such solutions are singular but compactly supported, somewhat akin to the compacton solutions studied in [Ros1] and references therein. For example, if we take the double layer shown in figure 7 and zero it on the left of the left singularity and on the right of the right singularity, we obtain a rather intersecting solution shown in figure 14 with a $\sin(\omega_1 x - \sqrt{\omega_1^2 + 1}\ t)$-like core, $\omega_1 = \frac{4\lambda_1^2 - 1}{4\lambda_1}$ for $-T < t < T$ for some T. Such solutions behave similarly to $\sin(\omega_1 x - \sqrt{\omega_1^2 + 1}\ t)$ in the sense that they exhibit similar properties like interference and diffraction, yet they are compactly supported; in the spirit of [Ros1], we shall call such solutions *basic compactons*.

Let $f(x,t)$ be a solution of the sine-Gordon equation describing a modulation akin to the ones illustrated in figures 8, 9, 11 and 12 and let $x_{s-}(t)$, $x_{s+}(t)$ denote correspondingly the first singularity on the left and the first singularity on the right from the domain of modulation. Due to the argument just given without any loss of generality, we may assume that $f(x,t) = 0$

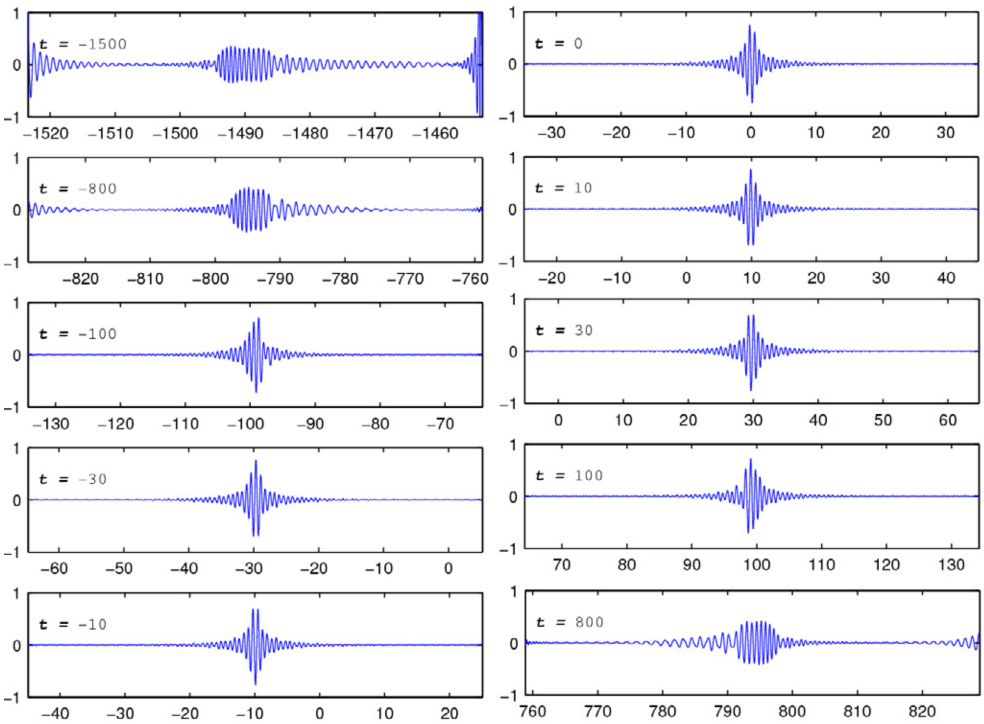

Figure 13. Time evolution of a modulated wave.

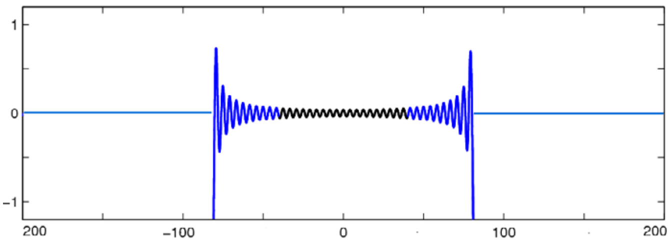

Figure 14. Snapshot of a singular compacton.

for $x < x_{s-}(t)$ and $x > x_{s+}(t)$, in which case the solution has the structure shown in figure 15 for $-T < t < T$ for some T. As t evolves from $-T$ to T, both $x_{s-}(t)$ and $x_{s+}(t)$ also evolve as does the form of $f(x, t)$ on the interval $x_{s-}(t) < x < x_{s+}(t)$. We call such a solution a *sine-Gordon universe.*

Sine-Gordon universes are only defined for $-T < t < T$, where T is a number whose exact value depends on the function $f(x, t)$. Once $|t| > T$, the singularities of the solution eventually catch up with the modulated profile destroying it. As shown in figures 11–13, the evolution of a sine-Gordon universe consists of two stages: the first 'contracting' stage $-T < t < 0$ at which the solution focuses and the second 'expanding' stage $0 < t < T$ at which the solution disperses. A typical sine-Gordon universe may be constructed as a nonlinear superposition of basic compactons; each basic compacton moves with a speed whose absolute value is less

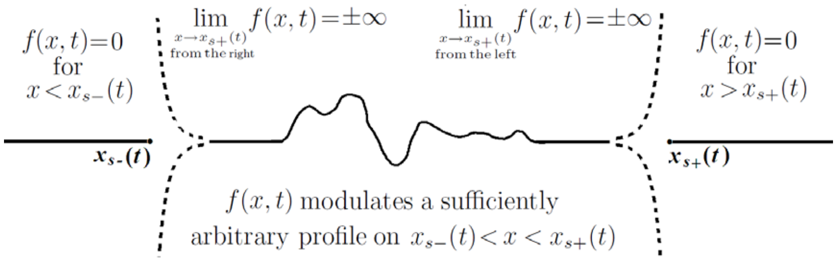

Figure 15. Snapshot of a sine-Gordon universe.

than 1. Sine-Gordon universes are comprised of basic compactons just like physical matter is comprised of elementary particles or solutions $\int_{-\infty}^{+\infty}\widehat{f}(\omega)\,\mathrm{e}^{\mathrm{i}(\omega x-\sqrt{\omega^2+1}\,t)}\,\mathrm{d}\omega$ of the linear Klein–Gordon equation $\frac{\partial^2 f}{\partial t^2}-\frac{\partial^2 f}{\partial x^2}+f=-\frac{\partial^2 f}{\partial\xi\partial\eta}+f=0$ are comprised of functions $\mathrm{e}^{\mathrm{i}(\omega x-\sqrt{\omega^2+1}\,t)}$. As far as applications to the real world problems are concerned, functions $\mathrm{e}^{\mathrm{i}(\omega x-\sqrt{\omega^2+1}\,t)}$ have no singularities but extend all the way to $\pm\infty$ while the basic compactons do have singularities but are compactly supported; we may say that the singularities of basic compactons are the price we pay for compact support just like the infinite spread of $\mathrm{e}^{\mathrm{i}(\omega x-\sqrt{\omega^2+1}\,t)}$ is the price we pay for their smoothness and lack of singularities.

We may introduce a sine-Gordon universe with particles moving with speed ± 1 by taking a superposition of harmonic breathers plus a sum of integer multiples of weak kinks, each weak kink moving with the speed ± 1. Just like the massless neutrinos of the real physics practically do not interact with physical matter, the weak kinks or their superpositions do not interact with the sine-Gordon matter.

We may also introduce a sine-Gordon universe with analogs of physical black holes by taking a superposition of harmonic breathers and imaginary kinks/antikinks, each imaginary kink/antikink being a sine-Gordon analog of a physical black hole.

Although the similarity of the solutions of the sine-Gordon equation to the physical models of the Universe is quite curious, it is doubtful it might be of use as the sine-Gordon equation is too simplistic to be considered a meaningful approximation of the Einstein equations of general relativity. What is of interest here though is what happens to the sine-Gordon universes as time goes to $\pm\infty$. Passing through the focusing point at $t=0$, the sine-Gordon universes do not collapse to a point but rather disperse into chaos losing any kind of meaningful structure. Rephrasing the words of [Lon1], one may describe the evolution of the sine-Gordon universes with the words 'chaos thou art, to chaos returnest' making us wonder whether the Big Bang theory should be replaced with the 'chaos thou art, to chaos returnest'-theory [Lon1].

5. Conclusion

In this paper, we discussed a class of explicit singular solutions of the sine-Gordon equation

$$\frac{\partial^2 f}{\partial\xi\,\partial\eta}=\sin f \qquad \text{sG}$$

given by (1.11); we call them harmonic breathers. Although as shown in section 1 harmonic breathers may be obtained as limiting cases of two-soliton solutions their structure is richer than that of solitons and other soliton-like solutions. Whereas the solitons and soliton-like solutions move like particles, the distinctive property of harmonic breathers is that they combine both particle- and wave-like motions: their singularities move like particles while the tails exhibit wave-like behavior. The latter is best explained by comparing sG with Klein–Gordon equation

$$\frac{\partial^2 f}{\partial\xi\partial\eta} = f. \tag{KG}$$

The most elementary solutions of KG are

$$\sin\left(2\lambda\gamma + 2\lambda\xi - \frac{1}{2\lambda}\eta\right), \tag{5.1}$$

and they serve as building blocks for solutions of KG in the form of Fourier sums

$$\sum_n \widetilde{f}(\lambda_n)\sin\left(2\lambda_n\gamma(\lambda_n) + 2\lambda_n\xi - \frac{1}{2\lambda_n}\eta\right)\Delta\lambda_n, \tag{5.2}$$

where $\lambda_1, \lambda_2, \lambda_3, \ldots$ are some positive numbers, $\gamma(\lambda)$, $\widetilde{f}(\lambda)$ are some functions and $\Delta\lambda_n = \lambda_{n+1} - \lambda_n$. Limits of Fourier sums with appropriately chosen parameters are solutions of KG in the form of Fourier integrals

$$\int_0^{+\infty} \widetilde{f}(\lambda)\sin\left(2\lambda\gamma(\lambda) + 2\lambda\xi - \frac{1}{2\lambda}\eta\right)\mathrm{d}\lambda. \tag{5.3}$$

Since, at least for small f, sG may be viewed as a perturbation of KG by the term $\sin f - f$ we may expect at least some properties of KG and its solutions to be carried over to sG and its solutions.

Indeed as shown in section 3, harmonic breathers may be viewed as the sG analogs of (5.1) whereas the tails of the solutions of sG given by (2.3) or the cores of the solutions of sG given by (2.3) subject to (2.7) may be viewed as the sG analogs of (5.2). As demonstrated in section 3 just like sufficiently generic solutions of KG may be approximated by Fourier sums (5.2), sufficiently generic solutions of sG may be approximated by the tails of solutions of sG given by (2.3) or by the cores of solutions given by (2.3) and (2.7). That said we may view formula (2.3) as nonlinear superposition of harmonic breathers given by the diagonal elements of (2.3) just like (5.2) is often viewed as linear superposition of solutions (5.1). Moreover, just like we construct solutions of KG in the form of Fourier integrals (5.3), we may construct analogous solutions of sG by taking limits of solutions given by (2.3) with appropriately chosen parameters; so defined solutions of sG may be viewed as nonlinear superpositions of infinitely many harmonic breathers with each harmonic breather providing only an infinitesimal contribution to the whole just like Fourier integrals (5.3) are viewed as linear superpositions of basic KG solutions (5.1) with each $\sin\left(2\lambda\gamma(\lambda) + 2\lambda\xi - \frac{1}{2\lambda}\eta\right)\mathrm{d}\lambda$ providing only an infinitesimal contribution to (5.3). Just like the phenomenon of linear interference guarantees the existence of KG solutions (5.3), the phenomenon of nonlinear interference given by (2.5) assures the existence of sG analogs of (5.3). It is rather remarkable as, unlike KG, sG is a nonlinear equation, yet it possess nonlinear structures similar to summation, integration and interference.

As remarkable as is the similarity in the behavior of the solutions of sG and KG is the difference between them given the fact that the KG is the linearization of sG near $f = 0$. One difference is that the harmonic breathers have singularities whereas (5.1) do not. Although viewed by some researchers as something to avoid, singularities usually indicate the boundaries of the validity of a mathematical model thus indicating where the mathematical model fails to describe the phenomenon it is intended to describe.

Another difference between the sG and KG is that the former also possesses soliton solutions whereas the latter does not; the place of harmonic breathers in the realm of solitons and soliton-like solutions of sG is discussed in section 1 whereas the interaction of harmonic breathers with solitons and soliton-like solutions of sG is given by (2.1).

The very existence of such diversity of solitons and soliton-like solutions of sG suggests adding them to harmonic breathers as building blocks or 'particles' to form more general solutions of sG. We did in section 4 where we considered solutions of sG which we called 'sin-Gordon universes'; they exhibit many properties of the physical Universe. One of them is the time expansion of sine-Gordon universes as $t \to +\infty$, yet it is not a time expansion from a big bang but a time expansion from chaos. Even though we do not claim any relationship of sine-Gordon universes with the physical Universe, section 4 does provide food for thought as to whether the physical Universe started from a big bang or from chaos like sine-Gordon universes.

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Appendix A. Derivation of the superposition formula for solitons and harmonic breathers

To obtain the formula, we consider a special case of (1.1)

$$f(x,t) = -2\mathrm{i}\ln\frac{\det(1+g)}{\det(1-g)} \pmod{2\pi}, \tag{A.1a}$$

where g is an $(2N+M)\times(2N+M)$ matrix with entries

$$g_{mn} = \frac{c_n}{\mu_n+\mu_m}\,\mathrm{e}^{2\mu_n\xi+\frac{1}{2\mu_n}\eta}, \qquad m,n = 1,2,\ldots,2N+M, \tag{A.1b}$$

and for $n = 1,2,\ldots,N$

$$\begin{aligned}
&\mu_{2n-1} = \kappa + \mathrm{i}\lambda_n, \qquad \mu_{2n} = \kappa - \mathrm{i}\lambda_n,\\
&c_{2n-1} = \frac{2\mathrm{i}\kappa(\kappa+\mathrm{i}\lambda_n)}{\lambda_n}\,\mathrm{e}^{2(\kappa\beta_n+\lambda_n\gamma_n\,\mathrm{i})}, \qquad c_{2n} = -\frac{2\mathrm{i}\kappa(\kappa-\mathrm{i}\lambda_n)}{\lambda_n}\,\mathrm{e}^{2(\kappa\beta_n-\lambda_n\gamma_n\mathrm{i})},\\
&\kappa > 0, \qquad \lambda_n \in \mathbb{R}, \qquad \kappa\beta_n \in \mathbb{R}, \qquad \lambda_n\gamma_n \in \mathrm{IR},
\end{aligned} \tag{A.1c}$$

while for $n = 2N+1, 2N+2, \ldots, 2N+M$

$$c_n = 2\mu_n\,\mathrm{e}^{2\mu_n\alpha_n+\varphi\mathrm{i}}. \tag{A.1d}$$

Note that here we do not require that for $n = 2N+1, 2N+2, \ldots, 2N+M$, $\mu_n > 0$, $\alpha_n \in \mathbb{R}$, $\varphi \in [0, 2\pi)$ as in (2.1c); these conditions are not required for the derivation of formula (2.1b) and are assumed only for further use in the text.

One may rewrite (1.1) as

$$f(x,t) = -2\mathrm{i}\ln\frac{\det G_+}{\det G_-} \pmod{2\pi}, \tag{A.2a}$$

where $G_{\pm}$ are $(2N+M)\times(2N+M)$ matrices of the form

$$G_{\pm} = \begin{pmatrix} G_{\pm bb} & G_{\pm bs} \\ G_{\pm sb} & G_{\pm ss} \end{pmatrix}, \tag{A.2b}$$

$G_{\pm bb}$ are $2N\times 2N$ matrices made of 2×2 blocks

$$\begin{pmatrix} G_{\pm bb,\ 2n-1\,2m-1} & G_{\pm bb,\ 2n-1\,2m} \\ G_{\pm bb,\ 2n\,2m-1} & G_{\pm bb,\ 2n\,2m} \end{pmatrix} = \begin{pmatrix} \delta_{mn} \pm \frac{2\mathrm{i}\kappa(\kappa+\mathrm{i}\lambda_n)}{\lambda_n(2\kappa+\mathrm{i}\lambda_n+\mathrm{i}\lambda_m)}\,\mathrm{e}^{2(B_n+\mathrm{i}\Gamma_n)} & \pm\frac{2\mathrm{i}\kappa(\kappa+\mathrm{i}\lambda_n)}{\lambda_n(2\kappa+\mathrm{i}\lambda_n-\mathrm{i}\lambda_m)}\,\mathrm{e}^{2(B_n+\mathrm{i}\Gamma_n)} \\ \mp\frac{2\mathrm{i}\kappa(\kappa-\mathrm{i}\lambda_n)}{\lambda_m(2\kappa-\mathrm{i}\lambda_n+\mathrm{i}\lambda_m)}\,\mathrm{e}^{2(B_n-\mathrm{i}\Gamma_n)} & \delta_{mn}\mp\frac{2\mathrm{i}\kappa(\kappa-\mathrm{i}\lambda_n)}{\lambda_m(2\kappa-\mathrm{i}\lambda_n-\mathrm{i}\lambda_m)}\,\mathrm{e}^{2(B_n-\mathrm{i}\Gamma_n)} \end{pmatrix} \tag{A.2c}$$

with $n, m = 1, 2, \ldots, N$; $G_{\pm bs}$ are $2N\times M$ matrices made of 2×1 blocks

$$\begin{pmatrix} G_{\pm bs,\ 2n-1\,m} \\ G_{\pm bs,\ 2n\,m} \end{pmatrix} = \begin{pmatrix} \pm\frac{2\mathrm{i}\kappa(\kappa+\mathrm{i}\lambda_n)}{\lambda_n(\kappa+\mathrm{i}\lambda_n+\mu_m)}\,\mathrm{e}^{2(B_n+\mathrm{i}\Gamma_n)} \\ \mp\frac{2\mathrm{i}\kappa(\kappa-\mathrm{i}\lambda_n)}{\lambda_n(\kappa-\mathrm{i}\lambda_n+\mu_m)}\,\mathrm{e}^{2(B_n-\mathrm{i}\Gamma_n)} \end{pmatrix} \tag{A.2d}$$

with $n = 1, 2, \ldots, N,\ m = 2N+1, 2N+2, \ldots, 2N+M$; $G_{\pm sb}$ are $M\times 2N$ matrices made of 1×2 blocks

$$(G_{\pm sb,\ n\,2m-1} \quad G_{\pm sb,\ n\,2m}) = \left(\pm\frac{2\mu_n}{\mu_n+\kappa+\mathrm{i}\lambda_m}\,\mathrm{e}^{2A_n} \quad \pm\frac{2\mu_n}{\mu_n+\kappa-\mathrm{i}\lambda_m}\,\mathrm{e}^{2A_n}\right) \tag{A.2e}$$

with $n = 2N+1, 2N+2, \ldots, 2N+M,\ m = 1, 2, \ldots, N$; $G_{\pm ss}$ are $M\times M$ matrices made of 1×1 blocks

$$G_{\pm ss,\ nm} = \delta_{nm} \pm \frac{2\mu_n}{\mu_n+\mu_m}\,\mathrm{e}^{2A_n} \tag{A.2f}$$

with $n, m = 2N+1, 2N+2, \ldots, 2N+M$ and

$$\begin{aligned} A_n &= \mu_n\alpha_n + \frac{\varphi_n}{2}\mathrm{i} + \mu_n\xi + \frac{1}{4\mu_n}\eta \\ B_n &= \kappa\left(\beta_n + \xi + \frac{1}{4\left(\kappa^2+\lambda_n^2\right)}\eta\right) \\ \Gamma_n &= \lambda_n\left(\gamma_n + \xi - \frac{1}{4\left(\kappa^2+\lambda_n^2\right)}\eta\right). \end{aligned} \tag{A.2g}$$

Multiplying $G_{\pm}$ on the left-hand side by

$$H_l = \mathrm{diag}\left\{\begin{pmatrix} \pm\frac{\mathrm{e}^{-\mathrm{i}\Gamma_1}}{\sqrt{\kappa}} & \frac{\mathrm{e}^{\mathrm{i}\Gamma_1}}{\sqrt{\kappa}} \\ \mp\mathrm{e}^{-\mathrm{i}\Gamma_1} & \mathrm{e}^{\mathrm{i}\Gamma_1} \end{pmatrix}, \ldots, \begin{pmatrix} \pm\frac{\mathrm{e}^{-\mathrm{i}\Gamma_N}}{\sqrt{\kappa}} & \frac{\mathrm{e}^{\mathrm{i}\Gamma_N}}{\sqrt{\kappa}} \\ \mp\mathrm{e}^{-\mathrm{i}\Gamma_N} & \mathrm{e}^{\mathrm{i}\Gamma_N} \end{pmatrix}, \underbrace{\sqrt{\kappa}, \ldots, \sqrt{\kappa}}_{M\ \text{diagonal entries}\ \sqrt{\kappa}}\right\}$$

and on the right-hand side by

$$H_r = -\frac{1}{2}\mathrm{diag}\left\{\begin{pmatrix} \pm\frac{\mathrm{e}^{\mathrm{i}\Gamma_1}}{\sqrt{\kappa}} & \mp\mathrm{e}^{\mathrm{i}\Gamma_1} \\ \frac{\mathrm{e}^{-\mathrm{i}\Gamma_1}}{\sqrt{\kappa}} & \mathrm{e}^{-\mathrm{i}\Gamma_1} \end{pmatrix}, \ldots, \begin{pmatrix} \pm\frac{\mathrm{e}^{\mathrm{i}\Gamma_N}}{\sqrt{\kappa}} & \mp\mathrm{e}^{\mathrm{i}\Gamma_N} \\ \frac{\mathrm{e}^{-\mathrm{i}\Gamma_N}}{\sqrt{\kappa}} & \mathrm{e}^{-\mathrm{i}\Gamma_N} \end{pmatrix}, \underbrace{\frac{1}{\sqrt{\kappa}}, \ldots, \frac{1}{\sqrt{\kappa}}}_{M\ \text{diagonal entries}\ \frac{1}{\sqrt{\kappa}}}\right\},$$

we obtain for small κ

$$f(x,t) = -2\mathrm{i}\ln\frac{\det K_{\kappa+}}{\det K_{\kappa-}} \quad (\mathrm{mod}\, 2\pi), \tag{A.3a}$$

where $K_{\kappa\pm}$ are $(2N+M)\times(2N+M)$ matrices of the form

$$K_{\kappa\pm} = \begin{pmatrix} K_{\kappa bb\pm} & K_{\kappa bs\pm} \\ K_{\kappa sb\pm} & K_{\kappa ss\pm} \end{pmatrix}, \tag{A.3b}$$

$K_{\kappa bb\pm}$ are $2N\times 2N$ matrices made up of 2×2 blocks

$$\begin{pmatrix} K_{\kappa bb\pm,\ 2n-1\,2m-1} & K_{\kappa bb\pm,\ 2n-1\,2m} \\ K_{\kappa bb\pm,\ 2n\,2m-1} & K_{\kappa bb,\ 2n\,2m} \end{pmatrix}$$
$$= \begin{pmatrix} -2\delta_{mn}\left[\frac{\mathrm{e}^{2B_n}-1}{\kappa} \pm \frac{\mathrm{e}^{2B_n}\sin 2\Gamma_n}{\lambda_n}\right] - 4(1-\delta_{mn})\left[\frac{\mathrm{e}^{2B_n+\mathrm{i}(\Gamma_n-\Gamma_m)}}{\mathrm{i}(\lambda_n-\lambda_m)+2\kappa}\right. & \\ \left. - \frac{\mathrm{e}^{2A-\mathrm{i}(\Gamma_n-\Gamma_m)}}{\mathrm{i}(\lambda_n-\lambda_m)-2\kappa} \pm \frac{\mathrm{e}^{2B_n+\mathrm{i}(\Gamma_n+\Gamma_m)}}{\mathrm{i}(\lambda_n+\lambda_m)+2\kappa} \pm \frac{\mathrm{e}^{2B_n-\mathrm{i}(\Gamma_n+\Gamma_m)}}{\mathrm{i}(\lambda_n+\lambda_m)}\right] + O(\kappa) & O(\sqrt{\kappa}) \\ O(\sqrt{\kappa}) & 4\delta_{mn} + O(\kappa) \end{pmatrix} \tag{A.3c}$$

with $n, m = 1, 2, \ldots, N$; $K_{\kappa bs\pm}$ are $2N\times M$ matrices made up of 2×1 blocks

$$\begin{pmatrix} K_{\kappa bs\pm,\ 2n-1\,m} \\ K_{\kappa bs\pm,\ 2n\,m} \end{pmatrix} = \begin{pmatrix} \frac{2\mathrm{i}(\kappa+\mathrm{i}\lambda_n)\,\mathrm{e}^{2B_n+\mathrm{i}\Gamma_n}}{\lambda_n(\kappa+\mathrm{i}\lambda_n+\mu_m)} \pm \frac{2\mathrm{i}(\kappa-\mathrm{i}\lambda_n)\,\mathrm{e}^{2B_n-\mathrm{i}\Gamma_n}}{\lambda_n(\kappa-\mathrm{i}\lambda_n+\mu_m)} \\ O(\sqrt{\kappa}) \end{pmatrix} \tag{A.3d}$$

with $n = 1, 2, \ldots, N,\ m = 2N+1, 2N+2, \ldots, 2N+M$; $K_{\kappa sb\pm}$ are $M\times 2N$ matrices made up of 1×2 blocks

$$(K_{\kappa sb\pm,\ n\,2m-1} \quad K_{\kappa sb\pm,\ n\,2m}) = \left(\frac{2\mu_n\,\mathrm{e}^{2A_n+\mathrm{i}\Gamma_m}}{\mu_n+\kappa+\mathrm{i}\lambda_n} \mp \frac{2\mu_n\,\mathrm{e}^{2A_n-\mathrm{i}\Gamma_m}}{\mu_n+\kappa-\mathrm{i}\lambda_n} \quad O(\sqrt{\kappa})\right) \tag{A.3e}$$

with $n = 2N+1, 2N+2, \ldots, 2N+M,\ m = 1, 2, \ldots, N$; $K_{\kappa ss\pm}$ are $M\times M$ matrices with entries

$$K_{\kappa ss\pm,\ nm} = \delta_{nm} \pm \frac{2\mu_n}{\mu_n+\mu_m}\,\mathrm{e}^{2A_n} \tag{A.3f}$$

with $n, m = 2N+1, 2N+2, \ldots, 2N+M$. Here $O(\sqrt{\kappa})$, $O(\kappa)$ denote correspondingly terms proportional to $\sqrt{\kappa}$ and κ.

As $\kappa \to 0$, the terms $O(\sqrt{\kappa}) \to 0$, $O(\kappa) \to 0$,

$$\begin{pmatrix} K_{\kappa bb\pm,\ 2n-12m-1} & K_{\kappa bb\pm,\ 2n-12m} \\ K_{\kappa bb\pm,\ 2n2m-1} & K_{\kappa bb,\ 2n2m} \end{pmatrix}$$
$$\to \begin{pmatrix} -4\delta_{mn}\left[\beta_n + \xi + \frac{1}{4\lambda_n^2}\eta\right] - 2(1-\delta_{mn})\frac{2\sin(\Gamma_n-\Gamma_m)}{\lambda_n-\lambda_m} \pm \frac{2\sin(\Gamma_n+\Gamma_m)}{\lambda_n+\lambda_m} & 0 \\ 0 & 4\delta_{nm} \end{pmatrix},$$

$$\begin{pmatrix} K_{\kappa bs\pm,\ 2n-1m} \\ K_{\kappa bs\pm,\ 2nm} \end{pmatrix} \to \begin{pmatrix} -\frac{2\,\mathrm{e}^{\mathrm{i}\Gamma_n}}{\mu_m+\mathrm{i}\lambda_n} \mp \frac{2\,\mathrm{e}^{-\mathrm{i}\Gamma_n}}{\mu_m-\mathrm{i}\lambda_n} \\ 0 \end{pmatrix},$$

$$(K_{\kappa sb\pm,\ n2m-1} \quad K_{\kappa sb\pm,\ n2m}) \to \left(\frac{2\mu_n\,\mathrm{e}^{2A_n+\mathrm{i}\Gamma_m}}{\mu_n+\mathrm{i}\lambda_n} \pm \frac{2\mu_n\,\mathrm{e}^{2A_n-\mathrm{i}\Gamma_m}}{\mu_n-\mathrm{i}\lambda_n} \quad 0\right),$$

entries of the columns and rows with even indices $2n$ become $4\delta_{mn}$ and hence can be dropped without affecting the values of the determinants, and formulas (A.3) become

$$f(x,t) = -2\mathrm{i}\ln\frac{\det K_{0+}}{\det K_{0-}} \quad (\mathrm{mod}\, 2\pi), \tag{A.4a}$$

where $K_{0\pm}$ is an $(N+M)\times(N+M)$ matrix with the entries

$$K_{0\pm\,nm} = \begin{cases} \delta_{mn}\left[\beta_n + \xi + \dfrac{1}{4\lambda_n^2}\eta\right] + (1-\delta_{mn})\dfrac{\sin(\Gamma_n-\Gamma_m)}{\lambda_n-\lambda_m} \pm \dfrac{\sin(\Gamma_n+\Gamma_m)}{\lambda_n+\lambda_m} \\ \quad \text{for } n,m = 1,2,\ldots,N, \\ \dfrac{1}{\mu_m^2+\lambda_n^2}\left[\dfrac{1\pm 1}{2}(\mu_m\cos\Gamma_n + \lambda_n\sin\Gamma_n) - \right. \\ \quad \left. \mathrm{i}\dfrac{1\mp 1}{2}(\lambda_n\cos\Gamma_n - \mu_m\sin\Gamma_n)\right] \\ \quad \text{for } n = 1,2,\ldots,N,\ m = N+1, N+2,\ldots,N+M, \\ -\dfrac{\mu_n\,\mathrm{e}^{2A_n}}{\mu_n^2+\lambda_m^2}\left[\dfrac{1\pm 1}{2}(\mu_n\cos\Gamma_m + \lambda_m\sin\Gamma_m) - \right. \\ \quad \left. \mathrm{i}\dfrac{1\mp 1}{2}(\lambda_m\cos\Gamma_m - \mu_n\sin\Gamma_m)\right] \\ \quad \text{for } n = N+1, N+2,\ldots,N+M,\ m = 1,2,\ldots,N, \\ -\dfrac{1}{4}\delta_{nm} \mp \dfrac{\mu_n}{2(\mu_n+\mu_m)}\,\mathrm{e}^{2A_n} \qquad \text{for } n,m = 2N+1, 2N+2,\ldots,2N+M. \end{cases} \tag{A.4b}$$

Introducing

$$K_{1\pm} = H_0^{-1} K_{0\pm} H_0, \tag{A.5}$$

where $H_0 = \mathrm{diag}\{\underbrace{1,1,\ldots,1}_{N\,\text{ones}},\ \mathrm{i}\sqrt{\mu_1}\,\mathrm{e}^{A_1},\ \mathrm{i}\sqrt{\mu_2}\,\mathrm{e}^{A_2},\ \ldots,\ \mathrm{i}\sqrt{\mu_M}\,\mathrm{e}^{A_M}\}$ we can rewrite (A.4) as

$$f(x,t) = -2\mathrm{i}\ln\frac{\det K_{1+}}{\det K_{1-}} \quad (\mathrm{mod}\ 2\pi), \tag{A.6a}$$

where $K_{1\pm}$ are $(N+M)\times(N+M)$ symmetric matrices with the entries

$$K_{1\pm\,nm} = \begin{cases} \delta_{mn}\left[\beta_n + \xi + \dfrac{1}{4\lambda_n^2}\eta\right] + (1-\delta_{mn})\dfrac{\sin(\Gamma_n-\Gamma_m)}{\lambda_n-\lambda_m} \pm \dfrac{\sin(\Gamma_n+\Gamma_m)}{\lambda_n+\lambda_m}, \\ \quad \text{for } n,m = 1,2,\ldots,N, \\ \dfrac{\sqrt{\mu_n}\,\mathrm{e}^{A_n}}{\mu_n^2+\lambda_m^2}\left[\dfrac{1\mp 1}{2}(\lambda_m\cos\Gamma_m - \mu_n\sin\Gamma_m) + \dfrac{\mathrm{i}\pm\mathrm{i}}{2}(\mu_n\cos\Gamma_m + \lambda_m\sin\Gamma_m)\right] \\ \quad \text{for } n = N+1, N+2,\ldots,N+M,\ m = 1,2,\ldots,N, \\ \text{the same as the previous line with } n \text{ and } m \text{ interchanged}, \\ \quad \text{for } n = 1,2,\ldots,N,\ m = N+1, N+2,\ldots,N+M, \\ -\dfrac{1}{4}\delta_{nm} \mp \dfrac{\sqrt{\mu_n\mu_m}}{2(\mu_n+\mu_m)}\,\mathrm{e}^{A_n+A_m} \qquad \text{for } n,m = 2N+1, 2N+2,\ldots,2N+M. \end{cases} \tag{A.6b}$$

One may further simplify (A.6) by multiplying K_{1+} both on the left-hand side and on the right-hand side by $H_1 = \mathrm{diag}\{\underbrace{1,1,\ldots,1}_{N\,\text{ones}},\ \underbrace{-\mathrm{i},\ -\mathrm{i},\ \ldots,\ -\mathrm{i}}_{M\,\mathrm{i}}\}$. That gives us

$$f(x,t) = -2\mathrm{i}\ln\frac{\det K_{+}}{\det K_{-}} \quad (\mathrm{mod}\ 2\pi), \tag{A.7a}$$

where $K_+ = H_1 K_{1+} H_1$, $K_- = K_{1+}$ are $(N+M)\times(N+M)$ symmetric matrices with the entries

$$K_{\pm\, nm} = \begin{cases} \delta_{mn}\left[\beta_n + \xi + \dfrac{1}{4\lambda_n^2}\eta\right] + (1-\delta_{mn})\dfrac{\sin(\Gamma_n - \Gamma_m)}{\lambda_n - \lambda_m} \pm \dfrac{\sin(\Gamma_n + \Gamma_m)}{\lambda_n + \lambda_m}, \\ \quad \text{for } n, m = 1, 2, \ldots, N, \\ \dfrac{\sqrt{\mu_n}\,\mathrm{e}^{A_n}}{\mu_n^2 + \lambda_m^2}\left[\dfrac{1\mp 1}{2}(\lambda_m\cos\Gamma_m - \mu_n\sin\Gamma_m) + \dfrac{1\pm 1}{2}(\mu_n\cos\Gamma_m + \lambda_m\sin\Gamma_m)\right] \\ \quad \text{for } n = N+1, N+2, \ldots, N+M,\ m = 1, 2, \ldots, N, \\ \text{the same as the previous line with } n \text{ and } m \text{ interchanged}, \\ \quad \text{for } n = 1, 2, \ldots, N,\ m = N+1, N+2, \ldots, N+M, \\ \pm\dfrac{1}{4}\delta_{nm} + \dfrac{\sqrt{\mu_n\mu_m}}{2(\mu_n+\mu_m)}\mathrm{e}^{A_n + A_m} \qquad \text{for } n, m = 2N+1, 2N+2, \ldots, 2N+M. \end{cases} \tag{A.7b}$$

Formulas (A.4) and (A.6) describe nonlinear superposition of N harmonic breathers and M kinks/antikinks, the latter in terms of symmetric matrices.

To obtain the formula for superposition of M kinks/antikinks, $\widetilde{M}$ weak kinks/antikinks and N harmonic breathers consider (1.7) with $M + \widetilde{M}$ kinks/antikinks

$$\widetilde{f}(x,t) = -2\mathrm{i}\ln\frac{\det\widetilde{K}_+}{\det\widetilde{K}_-} \pmod{2\pi}, \tag{A.8a}$$

$$\widetilde{K}_{\pm\, nm} = \begin{cases} \delta_{mn}\left[\beta_n + \xi + \dfrac{1}{4\lambda_n^2}\eta\right] + (1-\delta_{mn})\dfrac{\sin(\Gamma_n - \Gamma_m)}{\lambda_n - \lambda_m} \pm \dfrac{\sin(\Gamma_n + \Gamma_m)}{\lambda_n + \lambda_m}, \\ \quad \text{for } n, m = 1, 2, \ldots, N, \\ \dfrac{\sqrt{\mu_n}\,\mathrm{e}^{A_n}}{\mu_n^2 + \lambda_m^2}\left[\dfrac{1\mp 1}{2}(\lambda_m\cos\Gamma_m - \mu_n\sin\Gamma_m) + \dfrac{1\pm 1}{2}(\mu_n\cos\Gamma_m + \lambda_m\sin\Gamma_m)\right] \\ \quad \text{for } n = N+1, N+2, \ldots, N+M+\widetilde{M},\ m = 1, 2, \ldots, N, \\ \text{the same as the previous line with } n \text{ and } m \text{ interchanged}, \\ \quad \text{for } n = 1, 2, \ldots, N,\ m = N+1, N+2, \ldots, N+M+\widetilde{M}, \\ \pm\dfrac{1}{4}\delta_{nm} + \dfrac{\sqrt{\mu_n\mu_m}}{2(\mu_n+\mu_m)}\mathrm{e}^{A_n + A_m} \qquad \text{for } n, m = 2N+1, 2N+2, \ldots, 2N+M+\widetilde{M}, \end{cases} \tag{A.8b}$$

$$\varphi_n = \begin{cases} \dfrac{\pi}{2} & \text{for a real kink}, \\ \dfrac{\pi}{2} & \text{for a real antikink}, \end{cases} \qquad \text{for } n = N+M+1, \ldots, N+M+\widetilde{M}. \tag{A.8c}$$

pick a $j \in \{N+M+1, \ldots, N+M+\widetilde{M}\}$ and let $\mu_j \to +\infty$ while keeping α_j fixed. Then, $\det\widetilde{K}_\pm = \left[\pm\frac{1}{4} + \frac{1}{4}\mathrm{e}^{2A_j}\right]\det\widetilde{\widetilde{K}}_\pm$ + terms of order $o(\mathrm{e}^{2A_j})$, where $\widetilde{\widetilde{K}}_\pm$ are matrices obtained from $\widetilde{K}_\pm$ by deleting the jth row and jth column. Hence,

$$\begin{aligned} \lim_{\mu_j\to+\infty}\widetilde{f}(x,t) &= -2\mathrm{i}\ln\frac{\det\widetilde{\widetilde{K}}_+}{\det\widetilde{\widetilde{K}}_-} + \lim_{\mu_j\to+\infty}\left[-2\mathrm{i}\ln\frac{1+\mathrm{e}^{2A_j}}{-1+\mathrm{e}^{2A_j}}\right] \pmod{2\pi} \\ &= -2\mathrm{i}\ln\frac{\det\widetilde{\widetilde{K}}_+}{\det\widetilde{\widetilde{K}}_-} + \lim_{\mu_j\to+\infty}\left[-2\mathrm{i}\ln\frac{1+\mathrm{e}^{2A_j}}{1-\mathrm{e}^{2A_j}}\right] \pmod{2\pi} \\ &= -2\mathrm{i}\ln\frac{\det\widetilde{\widetilde{K}}_+}{\det\widetilde{\widetilde{K}}_-} + \begin{cases} 0, & \text{if} \quad \alpha_j + \xi < 0 \\ 4\varphi_j, & \text{if} \quad \alpha_j + \xi > 0 \end{cases} \pmod{2\pi}. \end{aligned}$$

Similarly, we obtain that if $\mu_j \to +0$ while keeping $\alpha_j = \frac{\widetilde{\alpha}_j}{\mu_j^2}$ with fixed $\widetilde{\alpha}_j$,

$$\lim_{\mu_j \to +\infty} \widetilde{f}(x,t) = -2\mathrm{i}\ln\frac{\det \widetilde{\widetilde{K}}_+}{\det \widetilde{\widetilde{K}}_-} + \begin{cases} 0, & \text{if} \quad \widetilde{\alpha}_j + \eta < 0 \\ 4\varphi_j, & \text{if} \quad \widetilde{\alpha}_j + \eta > 0 \end{cases} \quad (\mathrm{mod}\, 2\pi).$$

Taking limits as $\mu_j \to +\infty$ or $\mu_j \to +0$ for all $j \in \{N+M+1, \ldots, N+M+\widetilde{M}\}$, we obtain

$$\lim_{\substack{\text{all } \mu_j \text{ approach} \\ +\infty \text{ or } +0}} \widetilde{f}(x,t) = -2\mathrm{i}\ln\frac{\det K_+}{\det K_-} + f_{w1}(x-t) + f_{w2}(x+t) \tag{A.9}$$

where $K_\pm$ are the same as in (1.7*b*) or (2.1) and $f_{w1}(x-t)$, $f_{w2}(x+t)$are two step functions attaining values from the set $0, \pm 2\pi, \pm 4\pi, \ldots, \pm 2M\pi$ of multiples of 2π with the first function $f_{w1}(x-t)$ being a function of $x-t$ only while the second function $f_{w2}(x+t)$ is a function of $x+t$ only.

Formula (2.1) may also be obtained by utilizing the Darboux transform well described in [Mat1]. As shown in [And1], the superposition of M solitons may be written as

$$f(x,t) = \frac{2}{\mathrm{i}}\log\frac{\det\Phi}{\det\Psi}, \tag{A.10a}$$

where Φ and Ψ are $M \times M$ matrices

$$\Phi = \begin{pmatrix} \varphi_1 & \varphi_2 & \varphi_3 & \cdots \\ \mu_1\psi_1 & \mu_2\psi_2 & \mu_3\psi_3 & \cdots \\ \mu_1^2\varphi_1 & \mu_2^2\varphi_2 & \mu_3^2\varphi_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad \Psi = \begin{pmatrix} \psi_1 & \psi_2 & \psi_3 & \cdots \\ \mu_1\varphi_1 & \mu_2\varphi_2 & \mu_3\varphi_3 & \cdots \\ \mu_1^2\psi_1 & \mu_2^2\psi_2 & \mu_3^2\psi_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{A.10b}$$

$$\varphi_n = \mu_n \prod_{k\neq n}\frac{\mu_n+\mu_k}{\mu_n-\mu_k}\mathrm{e}^{-\mu_n\xi} + \frac{\mathrm{i}c_n}{2}\mathrm{e}^{\mu_n\xi} \qquad \psi_n = \mu_n \prod_{k\neq n}\frac{\mu_n+\mu_k}{\mu_n-\mu_k}\mathrm{e}^{-\mu_n\xi} - \frac{\mathrm{i}c_n}{2}\mathrm{e}^{\mu_n\xi}. \tag{A.10c}$$

Then,

$$1+g = \mathrm{diag}\left\{\mathrm{e}^{\mu_1\xi}\prod_{k\neq 1}(\mu_1-\mu_k), \ldots, \mathrm{e}^{\mu_M\xi}\prod_{k\neq 1}(\mu_M-\mu_k)\right\} F^{-1}\Phi$$
$$\times\, \mathrm{diag}\left\{\frac{1}{\mu_1\prod_{k\neq 1}(\mu_1+\mu_k)}, \ldots, \frac{1}{\mu_M\prod_{k\neq 1}(\mu_M+\mu_k)}\right\}, \tag{A.11a}$$

$$1-g = \mathrm{diag}\left\{\mathrm{e}^{\mu_1\xi}\prod_{k\neq 1}(\mu_1-\mu_k), \ldots, \mathrm{e}^{\mu_M\xi}\prod_{k\neq 1}(\mu_M-\mu_k)\right\} F^{-1}\Psi$$
$$\times\, \mathrm{diag}\left\{\frac{1}{\mu_1\prod_{k\neq 1}(\mu_1+\mu_k)}, \ldots, \frac{1}{\mu_M\prod_{k\neq 1}(\mu_M+\mu_k)}\right\}, \tag{A.11b}$$

where

$$F = \begin{pmatrix} 1 & 1 & \cdots, & 1 \\ \mu_1 & \mu_2 & \cdots & \mu_M \\ \mu_1^2 & \mu_2^2 & \cdots & \mu_M^2 \\ \vdots & \vdots & \cdots & \vdots \\ \mu_1^{M-1} & \mu_2^{M-1} & \cdots & \mu_M^{M-1} \end{pmatrix}$$

is the van der Monde matrix.

Once the relationships (A.11) are established one may easily switch between representations (A.10) and (1.1). Since formulas (2.1) are obtained as special limiting cases of (1.1), we obtain the relationship between (2.1) and its Darboux transform analog by taking appropriate limits of (A.11).

Appendix B. How to count the number of singular points for the solutions given by (2.1) and (2.3)

In this appendix, we will show how to count the number of singular points of the solutions of the sine-Gordon equation given by formulas (2.1) and (2.3). The value of t is assumed to be fixed but arbitrary and the dependence on t is suppressed in the notation.

Let us begin with the simpler case of the solutions given by (2.3). Define

$$rank_{\pm}(x) = \text{the dimension of the largest linear subspace on which } \mathcal{K}_{\pm} \text{ is positive definite} \quad \text{(B.1)}$$

and

$$h_{\pm}(x,t,v) = v\mathcal{K}_{\pm}v^T = \sum_{n,m=1}^{N}\mathcal{K}_{\pm nm}v_m v_n = \sum_{n=1}^{N}\left(\beta_n + \frac{4\lambda_n^2+1}{8\lambda_n^2}x - \frac{4\lambda_n^2-1}{8\lambda_n^2}t\right)v_n^2 + \sum_{\substack{n,m=1\\ n\neq m}}^{N}\frac{\sin(\Gamma_n-\Gamma_m)}{\lambda_n-\lambda_m}v_m v_n \pm \sum_{n,m=1}^{N}\frac{\sin(\Gamma_n+\Gamma_m)}{\lambda_n+\lambda_m}v_m v_n, \quad \text{(B.2)}$$

where $v = (v_1, v_2, \dots, v_N)$ is an N-dimensional real row vector and v^T is its transpose. Since for all

$$v,\ \frac{\partial h_{\pm}(x,t,v)}{\partial x} = \frac{\partial}{\partial x}\sum_{n,m=1}^{N}\mathcal{K}_{\pm nm}v_m v_n = \begin{cases}\sum_{n=1}^{N}(\cos\Gamma_n v_n)^2 + \left(\frac{\sin\Gamma_n}{2\lambda_n}v_n\right)^2 > 0, & \text{when the upper signs are taken,}\\ \sum_{n=1}^{N}(\sin\Gamma_n v_n)^2 + \left(\frac{\cos\Gamma_n}{2\lambda_n}v_n\right)^2 > 0, & \text{when the lower signs are taken,}\end{cases}$$

functions $rank_{\pm}(x)$ are increasing step functions assuming only integer values. But for $|x|$ sufficiently large, $h_{\pm}(x,t,v) \sim x\sum_{n=1}^{N}\frac{4\lambda_n^2+1}{4\lambda_n^2}v_n^2$ and thus

$$rank_{\pm}(x) = \begin{cases}0, & \text{for } x < 0 \text{ and } |x| \text{ sufficiently large,}\\ N, & \text{for } x > 0 \text{ and } |x| \text{ sufficiently large.}\end{cases}$$

Consequently $rank_{\pm}(x)$ may only assume values $0, 1, \dots, N$ and must make N unit jumps at some points denoted correspondingly by $x_{1\pm}(t), x_{2\pm}(t), \dots, x_{N\pm}(t)$, these are the singular points of $f(x, t)$ given by (2.3). For some values of t, some of the points $x_{1\pm}(t), x_{2\pm}(t), \dots, x_{N\pm}(t)$ may confluence.

Let us now consider the solutions of the sine-Gordon equation given by (2.1) with the requirement that

$$\begin{aligned}&\text{for all } n = N+1, N+2, \dots, N+M_1+M_2, && \mu_n > 0,\\ &\text{for } n = N+1, N+2, \dots, N+M_1, && \varphi_n = 0,\\ &\text{for } n = N+M_1+1, N+M_1+2, \dots, N+M_1+M_2, && \varphi_n = \frac{\pi}{2},\\ &\text{for } n = N+M_1+M_2+1, \dots, N+M_1+M_2+M_3, && \varphi_n = \frac{\pi}{4} \text{ or } \frac{5\pi}{4}.\end{aligned} \quad \text{(B.3)}$$

Conditions (B.3) essentially describe a superposition of N harmonic breathers, M_1 imaginary kinks, M_2 imaginary antikinks and M_3 real kinks/antikinks. Let $M = M_1 + M_2 + M_3$.

Define V to be the set of $(N + M)$-dimensional row-vectors $\boldsymbol{v} = (v_1, v_2, \ldots, v_N, v_{N+1}, v_{N+2}, \ldots, v_{N+M_1+M_2+M_3})$ with all v_n real; V_1 to be the subspace of V comprised of vectors $\boldsymbol{v} = (v_1, v_2, \ldots, v_N, v_{N+1}, \ldots, v_{N+M_1+M_2}, \underbrace{0, 0, \ldots, 0, 0}_{M_3\text{zeros}})$ and V_2 to be the subspace of V comprised of vectors $\boldsymbol{v} = (\underbrace{0, 0, \ldots, 0, 0}_{N+M_1+M_2 \text{ zeros}}, v_{N+M_1+M_2+1}, v_{N+M_1+M_2+2}, \ldots, v_{N+M_1+M_2+M_3})$. Also define $\widetilde{K_\pm} = H_\varphi K_\pm H_\varphi$ where $H_\varphi = \operatorname{diag}\left\{\underbrace{1, 1, \ldots, 1}_{N\text{ones}}, \mathrm{e}^{-\mathrm{i}\varphi_{N+1}}, \mathrm{e}^{-\mathrm{i}\varphi_{N+2}}, \ldots, \mathrm{e}^{-\mathrm{i}\varphi_{N+M_1+M_2}}\right\}$.

Conditions (B.3) assure that $\widetilde{K_\pm} V_1$ is real, i.e. the coordinates of $\widetilde{K_\pm}\boldsymbol{v}, \boldsymbol{v} \in V_1$ are real, but $\widetilde{K_\pm} V_2$ is complex non-real, i.e. the coordinates of $\widetilde{K_\pm}\boldsymbol{v}, \boldsymbol{v} \in V_1, \boldsymbol{v} \neq 0$ are not real; that in turn implies that exactly $N + M_1 + M_2$ eigenvalues of $\widetilde{K_\pm}$ are real and M_3 eigenvalues of $\widetilde{K_\pm}$ are complex nonzero. The singularities of solutions (2.1) are the points where the real eigenvalues of $\widetilde{K_\pm}$ become zero. Define

$$rank_\pm(x) = \text{the dimension of the largest linear subspace of } V \text{ on which } \widetilde{K_\pm} \text{ is positive definite} \tag{B.4}$$

and

$$\begin{aligned} h_\pm(x, t, \boldsymbol{v}) = \boldsymbol{v}\widetilde{K_\pm}\boldsymbol{v}^T &= \sum_{n,m=1}^{N+M_1+M_2+M_3} \widetilde{K}_{\pm nm} v_m v_n = \sum_{n=1}^{N} \left(\beta_n + \frac{4\lambda_n^2 + 1}{8\lambda_n^2}x - \frac{4\lambda_n^2 - 1}{8\lambda_n^2}t\right) v_n^2 \\ &+ \sum_{\substack{n,m=1\\ n\neq m}}^{N} \frac{\sin(\Gamma_n - \Gamma_m)}{\lambda_n - \lambda_m} v_m v_n \pm \sum_{n,m=1}^{N} \frac{\sin(\Gamma_n + \Gamma_m)}{\lambda_n + \lambda_m} v_m v_n \\ &\pm 0.25 \sum_{n=N+1}^{N+M_1+M_2+M_3} \mathrm{e}^{-2\mathrm{i}\varphi_n} v_n^2 + \sum_{n,m=N+1}^{N+M_1+M_2+M_3} \frac{\sqrt{\mu_n\mu_m}}{2(\mu_n + \mu_m)} \mathrm{e}^{A_n + A_m - \mathrm{i}\varphi_n - \mathrm{i}\varphi_m} v_m v_n \\ &+ (1 \mp 1) \sum_{m=N+1}^{N+M_1+M_2+M_3} \sum_{n=1}^{N} \frac{\sqrt{\mu_m}\, \mathrm{e}^{A_m - \mathrm{i}\varphi_m}}{\mu_m^2 + \lambda_n^2} (\lambda_m \cos\Gamma_n - \mu_m \sin\Gamma_n) v_m v_n \\ &+ (1 \pm 1) \sum_{m=N+1}^{M_1+M_2+M_3} \sum_{n=1}^{N} \frac{\sqrt{\mu_m}\, \mathrm{e}^{A_m - \mathrm{i}\varphi_m}}{\mu_m^2 + \lambda_n^2} (\mu_m \cos\Gamma_n + \lambda_n \sin\Gamma_n) v_m v_n \end{aligned} \tag{B.5}$$

where $\boldsymbol{v}^T$ is the transposed of $\boldsymbol{v}$.

Since

$$\operatorname{Im}[h_\pm(x, t, v)] = \pm\frac{1}{4} \sum_{n=N+M_1+M_2+1}^{N+M_1+M_2+M_3} v_n^2 \qquad \text{and}$$

$$\left[\frac{\partial \operatorname{Re}[h_\pm(x, t, \boldsymbol{v})]}{\partial x} = \sum_{n,m=1}^{N+M_1+M_2+M_3} \frac{\partial \widetilde{K}_{\pm nm}}{\partial x} v_m v_n \geqslant 0\right]$$

for all $\boldsymbol{v}$, functions $rank_{\pm}(x)$ are increasing step functions assuming only integer values. But

$$h_{\pm}(x,t,\boldsymbol{v}) \sim \begin{cases} x\sum_{n=1}^{N}\frac{4\lambda_n^2+1}{8\lambda_n^2}v_n^2 \pm \frac{1}{4}\sum_{n=N+1}^{N+M_1+M_2+M_3} \mathrm{e}^{-2\mathrm{i}\varphi_n}v_n^2 \\ \text{for } x<0 \text{ and } |x| \text{ sufficiently large,} \\ x\sum_{n=1}^{N}\frac{4\lambda_n^2+1}{8\lambda_n^2}v_n^2 + \frac{1}{4}\sum_{n=N+1}^{N+M_1+M_2} \mathrm{e}^{2A_n-2\mathrm{i}\varphi_n}v_n^2 \\ +\frac{1}{4}\sum_{n=N+M_1+M_2+1}^{N+M_1+M_2+M_3}\left(\mathrm{e}^{2A_n-2\mathrm{i}\varphi_n}\pm\mathrm{e}^{-2\mathrm{i}\varphi_n}\right)v_n^2, \\ \text{for } x>0 \text{ and } |x| \text{ sufficiently large.} \end{cases}$$

Consequently $rank_{+}(x)$ may only assume values $M_1, M_1+1, \ldots, N+M_1+M_2$ and must make $N+M_2$ unit jumps at some points denoted correspondingly by $x_{1+}(t), x_{2+}(t), \ldots, x_{N+M_2\,+}(t)$ and $rank_{-}(x)$ may only assume values $M_2, M_2+1, \ldots, N+M_1+M_2$ and must make $N+M_1$ unit jumps at some points denoted correspondingly by $x_{1-}(t), x_{2-}(t), \ldots, x_{N+M_1\,-}(t)$. The points $x_{1+}(t), x_{2+}(t), \ldots, x_{N+M_2\,+}(t)$, $x_{1-}(t), x_{2-}(t), \ldots, x_{N+M_1\,-}(t)$ are the $2N+M_1+M_2$ singular points of $f(x,t)$ given by (2.1). For some values of t, some of these points may confluence.

Appendix C. Proof of (2.5)

To prove (2.5) note that as $\lambda_m \to \lambda_{m-1}$

$$B_m \pm \frac{\sin 2\Gamma_m}{2\lambda_m} + B_{m-1} \pm \frac{\sin 2\Gamma_{m-1}}{2\lambda_{m-1}} - 2\mathcal{K}_{\pm\, m-1m} \to \lambda_m\beta_m + \lambda_{m-1}\beta_{m-1} - 2\,\frac{\mathrm{d}(\lambda_m\gamma(\lambda_m))}{\mathrm{d}\lambda_m}, \tag{C.1}$$

and

$$\frac{\left(B_{m-1}\pm\frac{\sin 2\Gamma_{m-1}}{2\lambda_{m-1}}\right)\left(B_m\pm\frac{\sin 2\Gamma_m}{2\lambda_m}\right)-\mathcal{K}_{\pm m-1\,m}^2}{B_m\pm\frac{\sin 2\Gamma_m}{2\lambda_m}+B_{m-1}\pm\frac{\sin 2\Gamma_{m-1}}{2\lambda_{m-1}}-2\mathcal{K}_{\pm\, m-1\,m}} \to = \widetilde{\beta_m}+\xi+\frac{1}{4\lambda_m^2}\eta \pm \frac{\sin 2\Gamma_{m-1}}{2\lambda_{m-1}}, \tag{C.2}$$

where

$$\widetilde{\beta}_{m-1} = \frac{\beta_m\;\beta_{m-1} - \left[\frac{\mathrm{d}(\lambda_m\gamma(\lambda_m))}{\mathrm{d}\lambda_m}\right]^2}{\beta_m+\beta_{m-1}-2\;\frac{\mathrm{d}(\lambda_m\gamma(\lambda_m))}{\mathrm{d}\lambda_m}}.$$

To simplify the proof of (2.5) assume, without loss of generality, that $m=N$. Then,

$$\lim_{\lambda_N\to\lambda_{N-1}}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln\det\mathcal{K}_{\pm} = \lim_{\lambda_N\to\lambda_{N-1}}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln\det\mathcal{C}\mathcal{K}_{\pm}\mathcal{C}^* = \lim_{\lambda_N\to\lambda_{N-1}}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln\det$$

$$\begin{pmatrix} B_1 & \mathcal{K}_{\pm 12} & \cdots & \mathcal{K}_{\pm 1N-2} & \mathcal{K}_{\pm 1N-1} & \mathcal{K}_{\pm 1N}-\mathcal{K}_{\pm 1N-1} \\ \mathcal{K}_{\pm 12} & B_2 & \cdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ \mathcal{K}_{\pm 1N-2} & \cdots & & B_{N-2} & \mathcal{K}_{\pm N-2N-1} & \mathcal{K}_{\pm N-2N}-\mathcal{K}_{N-2N-1} \\ \mathcal{K}_{\pm 1N-1} & -\frac{(\mathcal{K}_{\pm 1N}-\mathcal{K}_{\pm 1N-1})(\mathcal{K}_{\pm N-1N}-B_{N-1})}{B_N+B_{N-1}-2\mathcal{K}_{\pm N-1N}} & \cdots & & \frac{B_{N-1}B_N-\mathcal{K}_{\pm N-1N}^2}{B_N+B_{N-1}-2\mathcal{K}_{\pm N-1N}} & O \\ \mathcal{K}_{\pm 1N} & -\mathcal{K}_{\pm 1N-1} & \cdots & & \mathcal{K}_{\pm N-1N}-B_{N-1} & B_N+B_{N-1}-2\mathcal{K}_{\pm N-1N} \end{pmatrix}$$

where $\mathcal{C}^*$ is the adjoint of $\mathcal{C}$ given by

$$\mathcal{C} = \begin{pmatrix} & & & 0 & 0 \\ & I_{(N-2)\times(N-2)} & & \vdots & \vdots \\ & & & & 0 \\ 0 & \cdots & 0 & \frac{(\tau_N - \mathcal{D}_{\pm N-1N})}{\tau_N + B_{N-1} - 2\mathcal{K}_\pm N-1N} & \frac{(B_{N-1} - \mathcal{K}_{\pm N-1N})}{\tau_N + \tau_{N-1} - 2\mathcal{K}_{\pm N-1N}} \\ 0 & \cdots & 0 & -1 & 1 \end{pmatrix}$$

and $I_{(N-2)\times(N-2)}$ is the $(N-2)$-dimensional identity matrix. Taking limit as $\lambda_N \to \lambda_{N-1}$ and using (C.1), (C.2), we obtain (2.5).

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